Dyadic Green's functions for curved waveguides and cavities and their reformulation

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[1] Dyadic Green's functions (DGFs) for continuously curved waveguides are important for the feeding and radiation problems of cylindrically conformal slotted-waveguide arrays. The major difficulty in the construction of these DGFs in curved waveguides and cavities is that there are no entire-domain TE or TM modes with respect to the curving direction, while the longitudinal-section electric (LSE) and magnetic (LSM) modes do not have the complete orthogonality in terms of the dot product as required by the conventional Ohm-Rayleigh method as practiced in literature. Therefore, the conventional Ohm-Rayleigh method for constructing DGFs is not applicable to curved waveguides. In this work, the DGFs are constructed with the help of the Lorentz reciprocity theorem and the mode orthogonality based on the concept of power flow, and by adding the source singularity terms. To reduce the orders of singularity of DGFs in their application to waveguide walls, the common form of DGFs is then reformulated into a form convenient for numerical computation by both forward and backward derivation procedures. Finally, a general procedure is proposed for the reformulation of DGFs for common types of waveguides. The DGFs derived are applicable to problems with curved waveguide junctions, and coupling and radiating slots for conformal slotted-waveguide antennas. *INDEX TERMS:* 0619 Electromagnetics: Electromagnetic theory; 0624 Electromagnetics: Guided waves; 0609 Electromagnetics: Antennas; *KEYWORDS:* Green's functions, eigenvalues/eigenfunctions, curved waveguides, conformal slotted-waveguide antennas

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1. Introduction

[2] Continuously curved waveguides are used not only as waveguide bends but also as feeding or radiating guides for cylindrically conformal waveguide slot antenna arrays [*Fan*, 1995]. In analyzing these problems, one often uses the dyadic Green's functions (DGFs) for curved waveguides and cavities [*Fan*, 1995; *Bates*, 1969; *Mittra*, 1972]. Although many expressions of DGFs have been derived for various straight waveguides and cavities [*Collin*, 1960; *Tai*, 1994; *Rahmat-Samii*, 1975; *Tai and Rozenfeld*, 1976; *Wang*, 1978, 1982; *Kisliuk*, 1980; *Daniele*, 1982; *Pathak*, 1983; *Yang*, 1992; *Li et al.*, 1995], to our knowledge, no expressions of DGFs for curved waveguides and cavities are available in literature.

[3] The major difficulty in the construction of the DGFs for curved waveguides is that, unlike for straight waveguides, there are no entire-domain TE or TM modes with respect to the curving (also guiding) direction. There exist only the longitudinal-section electric modes (LSE) and longitudinal-section magnetic modes (LSM) in curved waveguides. However, these two sets of modes do not have the complete orthogonality in terms of the dot product. Furthermore, in contrast to straight waveguides, one cannot carry out the infinite volume integration in a curved waveguide as required by the dot-product orthogonality because a curved waveguide is truncated at the two ends. As a result, the conventional Ohm-Rayleigh method as practiced in literature is not applicable to the curved waveguides problems. Moreover, unlike common straight rectangular waveguides, the complete expressions of DGFs for curved guides cannot be constructed from the potential functions by solving the scalar wave equations.

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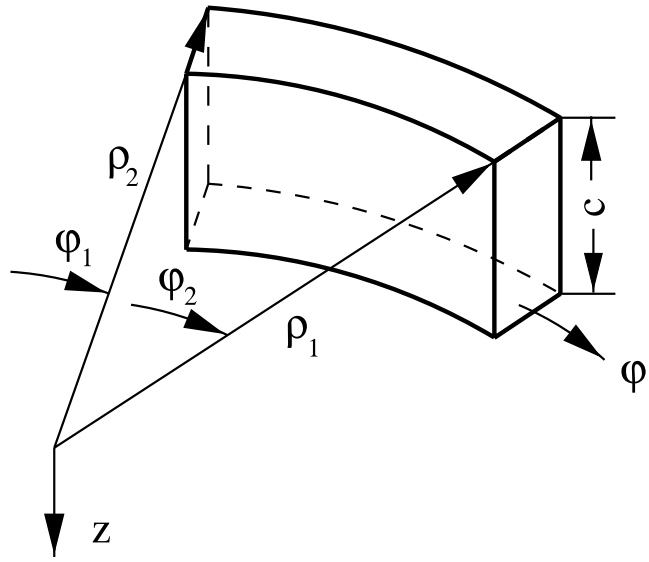

Figure 1. A curved waveguide and the coordinate system.

[4] Another issue to be addressed is the treatment of singularity in the DGFs. When applied to boundary integral equations, the common expressions of the DGFs are not convenient for numerical computation because of their higher-order singularity. In the past, much work has been done on the efficient numerical evaluation of DGFs in the source region for both unbounded space and waveguides and cavities in a volume integral form [*Wang*, 1982, 1991; *Pathak*, 1983; *Yang*, 1992; *Lee et al.*, 1980; *Yaghjian*, 1980; *Tai*, 1981; *Chew*, 1989, 1990; *Nachamkin*, 1990]. However, so far there is no report of a systematic approach to applying DGFs for waveguides and cavities to the boundary surfaces.

[5] In this paper, the DGFs for curved waveguides and cavities are constructed through the fields due to a point electric or magnetic source. The field expressions are derived with the help of the Lorentz reciprocity theorem and the LSE and LSM mode orthogonality based on the concept of power flow, and by adding the source singularity terms. The conventional form of DGFs is reformulated into a form convenient for numerical computation. Finally, a general procedure for reformulation of DGFs for general waveguides is proposed for an arbitrary waveguide.

2. Vector Wave Functions in Curved Waveguides

[6] Consider a curved waveguide (or cavity) of rectangular cross section with perfectly conducting walls, filled with homogeneous medium with the electrical parameters ϵ and μ, as depicted in Figure 1. With respect to a cylindrical coordinate system (ρ, φ, z), the inner waveguide walls are defined by $\rho = \rho_1$, ρ_2 and $z = 0$, c, and the two ends by $\varphi = \varphi_1$, φ_2. There are four possible cases for end conditions: both ends are matched (Case I) or shorted (Case II), the end $\varphi = \varphi_1$ is matched while the end $\varphi = \varphi_2$ is shorted (Case III), or vise versa (Case IV). A time dependence $\exp(j\omega t)$ is implied and suppressed in the following discussions.

[7] The eigenmodes in curved waveguides can be classified as either LSE modes or LSM modes [*Bates*, 1969; *Mittra*, 1972]. For the LSE modes, their vector wave functions are defined in terms of scalar wave functions $\psi^{\pm}_{\mu mn}$ as

$$\mathbf{M}^{\pm}_{\mu mn} = \nabla \times \hat{z}\psi^{\pm}_{\mu mn}, \tag{1}$$

$$\mathbf{N}^{\pm}_{\mu mn} = \frac{1}{k}\nabla \times \nabla \times \hat{z}\psi^{\pm}_{\mu mn}. \tag{2}$$

The scalar wave functions $\psi^{\pm}_{\mu mn}$ satisfy the same scalar Helmholtz equation and the same boundary conditions as those of H_z, and can be given by

$$\psi^{\pm}_{\mu mn} = B_{\mu_{mn}}(h_m\rho)\sin\left(\frac{m\pi z}{c}\right)\Phi^{\pm}_{\mu_{mn}}(\varphi), \tag{3}$$

in which

$$B_{\mu_{mn}}(h_m\rho) = J'_{\mu_{mn}}(h_m\rho_1)Y_{\mu_{mn}}(h_m\rho) - Y'_{\mu_{mn}}(h_m\rho_1)J_{\mu_{mn}}(h_m\rho), \tag{4}$$

$$\Phi^{\pm}_{\mu_{mn}}(\varphi) = \begin{cases} \begin{cases} \exp(+j\mu_{mn}\varphi) \\ \exp(-j\mu_{mn}\varphi) \end{cases} & \text{for Case I} \\ \begin{cases} \sin\mu_{mn}(\varphi-\varphi_1) \\ \sin\mu_{mn}(\varphi_2-\varphi) \end{cases} & \text{for Case II} \\ \begin{cases} \exp(+j\mu_{mn}\varphi) \\ \sin\mu_{mn}(\varphi_2-\varphi) \end{cases} & \text{for Case III} \\ \begin{cases} \sin\mu_{mn}(\varphi-\varphi_1) \\ \exp(-j\mu_{mn}\varphi) \end{cases} & \text{for Case IV.} \end{cases} \tag{5}$$

where the superscript $\pm$ denotes the upper and lower expressions, respectively. In the above formulas, k is the wavenumber, $h_m = \sqrt{k^2 - (m\pi/c)^2}(\Im m(h_m) \le 0)$, J and Y are Bessel functions of first and second kinds, the prime on the functions denotes derivative with respect to the argument, and the eigenvalues μ_{mn} are the roots of equation $B'_\mu(h_m\rho_2) = 0$.

[8] Similarly, for the LSM modes, the corresponding vector wave functions are defined in terms of scalar wave functions $\psi^{\pm}_{\epsilon mn}$ as

$$\mathbf{M}^{\pm}_{\epsilon mn} = \nabla \times \hat{z}\psi^{\pm}_{\epsilon mn}, \tag{6}$$

$$\mathbf{N}^{\pm}_{\epsilon mn} = \frac{1}{k}\nabla \times \nabla \times \hat{z}\psi^{\pm}_{\epsilon mn}. \tag{7}$$

The scalar wave functions $\psi^{\pm}_{\epsilon mn}$ satisfy the same scalar Helmholtz equation and the same boundary conditions as those of E_z, and are given by

$$\psi^{\pm}_{\epsilon mn} = C_{\epsilon_{mn}}(h_m\rho)\cos\left(\frac{m\pi z}{c}\right)\Phi^{\pm}_{\epsilon_{mn}}(\varphi), \tag{8}$$

$$C_{\epsilon_{mn}}(h_m\rho) = J_{\epsilon_{mn}}(h_m\rho_1)Y_{\epsilon_{mn}}(h_m\rho) - Y_{\epsilon_{mn}}(h_m\rho_1)J_{\epsilon_{mn}}(h_m\rho), \tag{9}$$

where the eigenvalues ϵ_{mn} are the roots of the equation $C_\epsilon(h_m\rho_2) = 0$, and $\Phi^{\pm}_{\epsilon mn}(\varphi)$ takes the same form as $\Phi^{\pm}_{\mu mn}(\varphi)$ in (5) except that μ_{mn} is replaced by ϵ_{mn} and the sine function by a cosine function.

[9] It can be shown that there is a one-to-one correspondence between these two sets of eigenmodes in curved waveguides and those in straight waveguides of the same cross section. However, in contrast to straight waveguides, the LSE(m,n) modes and LSM(m,n) modes in curved waveguides are not degenerate. Only when the radii of the curved walls become large, the eigenmode fields and the eigenvalues of curved waveguides approach those of straight waveguides.

[10] The symmetrical relations between the eigenfunctions **M** and **N** are

$$\mathbf{M}^{\pm}_{\substack{\epsilon\\ \mu}mn} = \frac{1}{k}\nabla \times \mathbf{N}^{\pm}_{\substack{\epsilon\\ \mu}mn}, \tag{10}$$

$$\mathbf{N}^{\pm}_{\substack{\epsilon\\ \mu}mn} = \frac{1}{k}\nabla \times \mathbf{M}^{\pm}_{\substack{\epsilon\\ \mu}mn}. \tag{11}$$

Similar relations exist for straight waveguides.

[11] Using the orthogonality relations of trigonometric functions and the radial functions in curved waveguides given by Appendix A, we can show that the above vector wave functions have the following orthogonal relations:

$$\int_{\rho_1}^{\rho_2}\int_0^c \left(\mathbf{M}^{\pm}_{\mu mn} \times \mathbf{N}^{\pm}_{\epsilon m'n'}\right)\cdot\hat{\varphi}d\rho\, dz = \int_{\rho_1}^{\rho_2}\int_0^c \left(\mathbf{N}^{\pm}_{\mu mn} \times \mathbf{M}^{\pm}_{\epsilon m'n'}\right)\cdot\hat{\varphi}d\rho\, dz = 0, \tag{12}$$

$$\int_{\rho_1}^{\rho_2}\int_0^c \left(\mathbf{M}^{\pm}_{\mu mn} \times \mathbf{N}^{\mp}_{\mu m'n'}\right)\cdot\hat{\varphi}d\rho\, dz = -\frac{1+\delta_{0m}}{2}\frac{h_m^2 c}{k}\delta_{mm'}\delta_{nn'}N_{\mu_{mn}}\Phi^{\pm'}_{\mu_{mn}}(\varphi)\Phi^{\mp}_{\mu_{mn}}(\varphi), \tag{13}$$

$$\int_{\rho_1}^{\rho_2}\int_0^c \left(\mathbf{M}^{\pm}_{\epsilon mn} \times \mathbf{N}^{\mp}_{\epsilon m'n'}\right)\cdot\hat{\varphi}d\rho\, dz = -\frac{1+\delta_{0m}}{2}\frac{h_m^2 c}{k}\delta_{mm'}\delta_{nn'}N_{\epsilon_{mn}}\Phi^{\pm'}_{\epsilon_{mn}}(\varphi)\Phi^{\mp}_{\epsilon_{mn}}(\varphi), \tag{14}$$

where $\delta_{pq} = 1$ for $p = q$, and $\delta_{pq} = 0$ for $p \neq q$. The mode normalization constants $N_{\mu_{mn}}$ and $N_{\epsilon_{mn}}$ have the expressions in integral form:

$$N_{\mu_{mn}} = \int_{\rho_1}^{\rho_2}\frac{1}{\rho}\left[B_{\mu_{mn}}(h_m\rho)\right]^2 d\rho, \tag{15}$$

$$N_{\epsilon_{mn}} = \int_{\rho_1}^{\rho_2}\frac{1}{\rho}\left[C_{\mu_{mn}}(h_m\rho)\right]^2 d\rho, \tag{16}$$

and in differential form:

$$N_{\mu_{mn}} = \frac{1}{2\mu_{mn}}\left\{h_m\rho B_{\mu_{mn}}(h_m\rho)\frac{\partial B_{\mu_{mn}}(h_m\rho)}{\partial\mu_{mn}}\right\}_{\rho=\rho_1}^{\rho_2}, \tag{17}$$

$$N_{\epsilon_{mn}} = \frac{-1}{2\epsilon_{mn}}\left\{h_m\rho C'_{\epsilon_{mn}}(h_m\rho)\frac{\partial C_{\epsilon_{mn}}(h_m\rho)}{\partial\epsilon_{mn}}\right\}_{\rho=\rho_1}^{\rho_2}. \tag{18}$$

Note that the above orthogonality is based on the concept of power flow, rather than the dot-product as required in the conventional Ohm-Rayleigh method. The above vector wave functions also form a complete set in the source-freeregion. Hence the fields outside the source

region can be uniquely expanded in terms of these functions.

[12] In passing, we point out that for lossless curved waveguides, the eigenvalues, μ_{mn} and ϵ_{mn}, are real for only the dominant mode or a few lower-order modes, and are imaginary forthe higher cutoff modes. The eigenfunctions involve Bessel functions and modified Bessel functions of imaginary orders. The accurate numerical methods for computing these functions have been given by *Fan and Yang* [1994a, 1994b, 1995].

3. Dyadic Green's Functions for Curved Waveguides and Cavities

3.1. Magnetic-Source Dyadic Green's Functions

[13] The magnetic-source electric-field DGF $\overline{\mathbf{G}}_E^{(m)}(\mathbf{r}, \mathbf{r}')$ and magnetic-source magnetic-field DGF $\overline{\mathbf{G}}_H^{(m)}(\mathbf{r}, \mathbf{r}')$ satisfy the following partial differential equations and boundary conditions, respectively

$$\left(\nabla \times \nabla \times -k^2\right)\overline{\mathbf{G}}_E^{(m)}(\mathbf{r}, \mathbf{r}') = -\nabla \times \overline{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}') \quad (19)$$

$$\hat{n} \times \overline{\mathbf{G}}_E^{(m)}(\mathbf{r}, \mathbf{r}') = 0 \quad \text{on the walls,} \quad (20)$$

and

$$\left(\nabla \times \nabla \times -k^2\right)\overline{\mathbf{G}}_H^{(m)}(\mathbf{r}, \mathbf{r}') = -j\omega\epsilon\overline{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}'), \quad (21)$$

$$\hat{n} \times \nabla \times \overline{\mathbf{G}}_H^{(m)}(\mathbf{r}, \mathbf{r}') = 0 \quad \text{on the walls.} \quad (22)$$

Note that for typesetting convenience (19) and (20) are written in the operator form, and the corresponding common form is givenby *Tai* [1994].

[14] The solutions of (19)–(22) can be written in the form [*Yang*, 1992]

$$\overline{\mathbf{G}}_E^{(m)}(\mathbf{r}, \mathbf{r}') = \sum_j \mathbf{E}^{(m)}\left(\mathbf{r}, \mathbf{r}'; \hat{u}'_j\right)\hat{u}'_j, \quad (23)$$

$$\overline{\mathbf{G}}_H^{(m)}(\mathbf{r}, \mathbf{r}') = \sum_j \mathbf{H}^{(m)}\left(\mathbf{r}, \mathbf{r}'; \hat{u}'_j\right)\hat{u}'_j, \quad (24)$$

where $\mathbf{r}'$ is the source location of a unit point magnetic dipole in the $\hat{u}'_j$-direction $\hat{u}'_j$ (j = 1,2,3 in an orthogonal coordinate system), and $\mathbf{E}^{(m)}(\mathbf{r}, \mathbf{r}'; \hat{u}'_j)$ and $\mathbf{H}^{(m)}(\mathbf{r}, \mathbf{r}'; u'_j)$ are the electric and magnetic fields at the field point $\mathbf{r}$ excited by this point magnetic dipole.

[15] In the following, we first determine the fields outside the source point by applying the Lorentz reciprocity theorem, and then add the source singularity-terms to the field expressions.

[16] For a unit point magnetic dipole source, the magnetic current density is $\mathbf{J}_m = \hat{m}'\delta(\mathbf{r} - \mathbf{r}')$. The excited fields in a source-free region can be expanded in terms of the vector eigenfunctions defined in the previous section, as follows

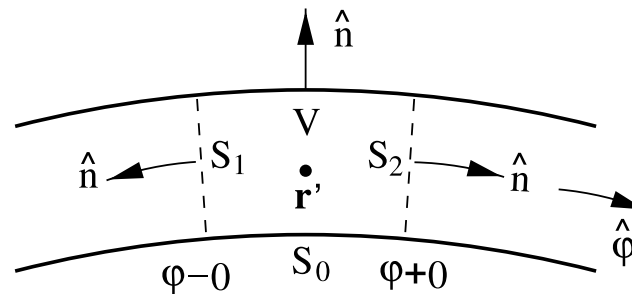

Figure 2. Division of integration region.

$$\mathbf{E}^{(m)\pm} = \sum_m \sum_n \left[\alpha_{mn}^{\pm}\mathbf{M}_{\mu mn}^{\pm} + \beta_{mn}^{\pm}\mathbf{N}_{\epsilon mn}^{\pm}\right]\varphi \gtrless \varphi', \quad (25)$$

$$\mathbf{H}^{(m)\pm} = \frac{1}{j\omega\mu}\sum_m \sum_n \left[\alpha_{mn}^{\pm}\mathbf{N}_{\mu mn}^{\pm} + \beta_{mn}^{\pm}\mathbf{M}_{\epsilon mn}^{\pm}\right]\varphi \gtrless \varphi', \quad (26)$$

where $\alpha_{mn}^{\pm}$ and $\beta_{mn}^{\pm}$ are coefficients to be determined.

[17] To obtain the expansion coefficients $\alpha_{mn}^{\pm}$ and $\beta_{mn}^{\pm}$ in (25) and (26), we now apply Lorentz reciprocity theorem [*Collin*, 1960] to a small volume enclosing the source. Let $\mathbf{E}^{(2)}$ and $\mathbf{H}^{(2)}$ be the fields produced by the magnetic current, $\mathbf{J}_m^{(2)} = \mathbf{J}_m$, and $\mathbf{E}^{(1)}$ and $\mathbf{H}^{(1)}$ be the eigenmode fields defined in the previous section. Consider the volume V bounded by the waveguide walls and two cross-sectional plane $\varphi = \varphi' \pm 0$, as shown in Figure 2. Now if Lorentz reciprocity theorem [*Collin*, 1960] is applied to this volume, the surface integral on S_0 vanishes because of the boundary condition, i.e., $\hat{n} \times \mathbf{E}^{(1)} = \hat{n} \times \mathbf{E}^{(2)} = 0$ on the walls. Also note that $\hat{n} = \hat{\varphi}$ on S_1, $\hat{n} = -\hat{\varphi}$ on S_2, and $\mathbf{J}_e^{(1)} = \mathbf{J}_e^{(2)} = \mathbf{J}_m^{(1)} = 0$, then the Lorentz reciprocity reduces to

$$\iint_{S_1} \left\{\mathbf{E}^{(1)} \times \left[\mathbf{H}^{(m)-} - \mathbf{H}^{(m)+}\right] - \left[\mathbf{E}^{(m)-} - \mathbf{E}^{(m)+}\right] \times \mathbf{H}^{(1)} \cdot \hat{\phi}\, dS = \iiint_V \mathbf{J}_m \cdot \mathbf{H}^{(1)} dV. \quad (27)$$

In the above equation, taking

$$\mathbf{E}^{(1)} = \mathbf{M}_{\mu mn}^{\pm}, \qquad \mathbf{H}^{(1)} = \frac{1}{j\omega\mu}\mathbf{N}_{\mu mn}^{\pm}, \quad (28)$$

and using the orthogonal relations given in the previous section, we obtain $\alpha_{mn}^{\pm}$ in (25) as

$$\alpha_{mn}^{\pm} = b_{mn}\mathbf{N}_{\mu mn}^{\mp}(\mathbf{r}') \cdot \hat{\mathbf{m}}', \quad (29)$$

where

$$b_{mn} = -\frac{2}{1 + \delta_{0m}} \frac{k}{c h_m^2 N_{\mu_{mn}} \Delta_{\mu_{mn}}}, \quad (30)$$

$$\Delta_{\mu_{mn}} = \Phi^{+\prime}_{\mu_{mn}}(\varphi')\,\Phi^{-}_{\mu_{mn}}(\varphi') - \Phi^{-\prime}_{\mu_{mn}}(\varphi')\,\Phi^{+}_{\mu_{mn}}(\varphi'). \tag{31}$$

[18] Similarly, by letting

$$\mathbf{E}^{(1)} = \mathbf{M}^{\pm}_{\epsilon mn}, \qquad \mathbf{H}^{(1)} = \frac{1}{j\omega\mu}\mathbf{N}^{\pm}_{\epsilon mn}, \tag{32}$$

we also arrive at

$$\beta^{\pm}_{mn} = a_{mn}\mathbf{N}^{\mp}_{\epsilon mn}(\mathbf{r}')\cdot\hat{\mathbf{m}}', \tag{33}$$

where

$$a_{mn} = -\frac{2}{1+\delta_{0m}}\frac{k}{ch_m^2 N_{\epsilon_{mn}}\Delta_{\epsilon_{mn}}}, \tag{34}$$

$$\Delta_{\epsilon mn} = \Phi^{+\prime}_{\epsilon_{mn}}(\varphi')\,\Phi^{-}_{\epsilon_{mn}}(\varphi') - \Phi^{-\prime}_{\epsilon_{mn}}(\varphi')\,\Phi^{+}_{\epsilon_{mn}}(\varphi'). \tag{35}$$

The constants $\Delta_{\mu_{mn}}$ and $\Delta_{\epsilon_{mn}}$ have the following explicit expressions for different end conditions,

$$\Delta_{\mu_{mn}} = \begin{cases} 2j\,\mu_{mn} & \text{for Case I} \\ \mu_{mn}\,\sin\mu_{mn}(\varphi_2-\varphi_1) & \text{for Case II} \\ \mu_{mn}\,\exp(j\mu_{mn}\varphi_2) & \text{for Case III} \\ \mu_{mn}\,\exp(-j\mu_{mn}\varphi_2) & \text{for Case IV} \end{cases}, \tag{36}$$

and

$$\Delta_{\epsilon_{mn}} = \begin{cases} 2j\,\epsilon_{mn} & \text{for Case I} \\ \epsilon_{mn}\,\sin\epsilon_{mn}(\varphi_2-\varphi_1) & \text{for Case II} \\ \epsilon_{mn}\,\exp(j\epsilon_{mn}\varphi_2) & \text{for Case III} \\ \epsilon_{mn}\,\exp(-j\epsilon_{mn}\varphi_2) & \text{for Case IV} \end{cases}. \tag{37}$$

[19] Note that equations (25) and (26) apply to a source-free region only. To arrive at expressions also valid at the source region, we turn to the problem of the source-singularity terms. In the work of *Pathak* [1983], the expressions of the source-singularity terms for the electric and magnetic fields due to a point electric source $\mathbf{J} = \hat{e}'\delta\,(\mathbf{r}-\mathbf{r}')$ are derived. By invoking the duality principle, we obtain the expressions of the electric and magnetic fields excited by a point magnetic source as follows,

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\pm}(\mathbf{r}), \quad \mathbf{H}(\mathbf{r}) = \mathbf{H}^{\pm}(\mathbf{r}) - \frac{\hat{p}\hat{p}}{j\omega\mu}\delta(\mathbf{r}-\mathbf{r}')\cdot\hat{m}', \quad p \gtrless p', \tag{38}$$

where $\hat{p}$ is the propagation direction of the vector wave functions with which the fields, $\mathbf{E}^{\pm}(\mathbf{r})$ and $\mathbf{H}^{\pm}(\mathbf{r})$, are constructed. The second term in $\mathbf{H}(\mathbf{r})$ involving $\delta(\mathbf{r}-\mathbf{r}')$ function is referred to as the source-singularity term, and it is irrotational. By examining the discontinuity of the fields on both sides of Maxwell's equations in scalar form, we also readily obtain the above expression for the source-singularity term, as discussed by *Yang* [1992].

[20] By taking $\hat{p} = \hat{\varphi}'$ and $\hat{m}' = \hat{u}'_j = \hat{\rho}', \hat{\varphi}'$ and $\hat{z}'$, respectively, and substituting the expressions of $\mathbf{E}^{(m)}$ and $\mathbf{H}^{(m)}$ derived into (1) and (2), we finally obtain

$$\overline{\mathbf{G}}^{(m)}_E(\mathbf{r},\mathbf{r}') = \sum_m\sum_n\left[a_{mn}\mathbf{N}^{\pm}_{\epsilon mn}(\mathbf{r})\mathbf{M}^{\mp}_{\epsilon mn}(\mathbf{r}') + b_{mn}\mathbf{M}^{\pm}_{\mu mn}(\mathbf{r})\mathbf{N}^{\mp}_{\mu mn}(\mathbf{r}')\right] \quad \varphi \gtrless \varphi', \tag{39}$$

$$\overline{\mathbf{G}}^{(m)}_H(\mathbf{r},\mathbf{r}') = \frac{-k}{j\omega\mu}\sum_m\sum_n\left[a_{mn}\mathbf{M}^{\pm}_{\epsilon mn}(\mathbf{r})\mathbf{M}^{\mp}_{\epsilon mn}(\mathbf{r}') + b_{mn}\mathbf{N}^{\pm}_{\mu mn}(\mathbf{r})\mathbf{N}^{\mp}_{\mu mn}(\mathbf{r}')\right] + \hat{\varphi}\hat{\varphi}'\delta(\mathbf{r}-\mathbf{r}')/(-j\omega\mu) \quad \varphi \gtrless \varphi', \tag{40}$$

where a_{mn} and b_{mn} are given by (30) and (34), respectively.

[21] It can be seen that the above relatively simple procedure, unlike the conventional Ohm-Rayleigh method, does not involve the infinite integrals in spectrum domain and the vector eigenfunction expansions of dyadic δ-functions. Moreover, the expressions of $\overline{\mathbf{G}}^{(m)}_E$ and $\overline{\mathbf{G}}^{(m)}_H$ are obtained simultaneously without additional derivation. Using the generalized function theory, we can show $\overline{\mathbf{G}}^{(m)}_H = (1/j\omega\epsilon)\nabla\times\overline{\mathbf{G}}^{(m)}_H$, similar to the relation of **E** and **H**.

3.2. Electric-Source Dyadic Green's Functions

[22] The electric-source electric-field DGF $\overline{\mathbf{G}}^{(e)}_H$ and electric-source magnetic-field DGF $\overline{\mathbf{G}}^{(e)}_H(\mathbf{r},\mathbf{r}')$ satisfy the following partial differential equations and boundary conditions, respectively

$$\left(\nabla\times\nabla\times -k^2\right)\overline{\mathbf{G}}^{(e)}_E(\mathbf{r},\mathbf{r}') = -j\omega\,\mu\bar{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}'), \tag{41}$$

$$\hat{n}\times\overline{\mathbf{G}}^{(e)}_E(\mathbf{r},\mathbf{r}') = 0 \quad \text{on the walls}, \tag{42}$$

and

$$\left(\nabla\times\nabla\times -k^2\right)\overline{\mathbf{G}}^{(e)}_H(\mathbf{r},\mathbf{r}') = \nabla\times\bar{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}'), \tag{43}$$

$$\hat{n}\times\nabla\times\overline{\mathbf{G}}^{(e)}_H(\mathbf{r},\mathbf{r}') = 0 \quad \text{on the walls}, \tag{44}$$

[23] Using a procedure similar to that for a magnetic source, we obtain,

$$\overline{\mathbf{G}}^{(e)}_E(\mathbf{r},\mathbf{r}') = \frac{-k}{j\omega\epsilon}\sum_m\sum_n\left[a_{mn}\mathbf{N}^{\pm}_{\epsilon mn}(\mathbf{r})\mathbf{N}^{\mp}_{\epsilon mn}(\mathbf{r}') + b_{mn}\mathbf{M}^{\pm}_{\mu mn}(\mathbf{r})\mathbf{M}^{\mp}_{\mu mn}(\mathbf{r}')\right] + \hat{\varphi}\hat{\varphi}'\delta(\mathbf{r}-\mathbf{r}')/(-j\omega\epsilon) \quad \varphi \gtrless \varphi', \tag{45}$$

$$\overline{\mathbf{G}}_H^{(e)}(\mathbf{r},\mathbf{r}') = -\sum_m \sum_n \left[a_{mn}\mathbf{M}_{\epsilon mn}^{\pm}(\mathbf{r})\mathbf{N}_{\epsilon mn}^{\mp}(\mathbf{r}') + b_{mn}\mathbf{N}_{\mu mn}^{\pm}(\mathbf{r})\mathbf{M}_{\mu mn}^{\mp}(\mathbf{r}')\right] \quad \varphi \gtrless \varphi', \tag{46}$$

where a_{mn} and b_{mn} are given by (30) and (34), respectively. It can also be confirmed that $\overline{\mathbf{G}}_H^{(e)} = (-1/j\omega\mu)\nabla\times\overline{\mathbf{G}}_H^{(e)}$. With the help of duality principle and taking account of the boundary conditions on the walls, we can also obtain the above expressions from those of the magnetic-source dyadic Green's functions.

4. Reformulation of Dyadic Green's Functions

[24] When the DGFs derived above are applied to the boundary integral equation, there exist two main computational difficulties if the source and field points coincide: one is the treatment of the δ-function, and the other is their higher-order singularities. To overcome these difficulties, in this section, we first introduce the eigenmode expansions of the source-singularity terms, then derive a new form of the DGF on the curved waveguide walls using two different procedures, namely the forward and backward derivations, based on the theory of generalized functions. The backward derivation procedure is finally extended to an arbitrary waveguide in an orthogonal curvilinear coordinate system.

4.1. Expansion of the Source-Singularity Terms

[25] It is well known that in curved waveguides, there are no pure TE and TM modes valid for the whole waveguide region. However, because of the localized effect of δ-function, the unit vector $\hat{p}$ in the source-point term $\hat{p}\hat{p}'\delta(\mathbf{r}-\mathbf{r}')$ for a given $\hat{p}$-propagating direction can be considered as a constant vector in the neighborhood of the source point. We first introduce the localized vector eigenfunctions, $\mathbf{M}_\epsilon^{(E)}$, $\mathbf{N}_\epsilon^{(E)}$, $\mathbf{M}_\mu^{(H)}$, and $\mathbf{N}_\mu^{(H)}$, respectively corresponding to TE and TM modes with the $\hat{p}$-direction. Then we define two new vector functions $\mathrm{L}_\epsilon^{(E)} = \nabla\psi_\epsilon^{(E)}$ and $\mathrm{L}_\mu^{(H)} = \nabla\psi_\mu^{(H)}$. Note that the **M**-functions do not have $\hat{p}$-components, while the $\hat{p}$-components of $\mathbf{N}^{(E)}$ and $\mathrm{L}^{(E)}$ or $\mathbf{N}^{(H)}$ and $\mathrm{L}^{(H)}$ have the same types of expansion functions. Therefore, the δ-function in the source-singularity terms for magnetic field and electric field can be expanded, respectively, as follows,

$$\delta_H(\mathbf{r}-\mathbf{r}') = \sum_\mu A_\mu \frac{1}{g_p}\frac{\partial\psi_\mu^{(H)}}{\partial p}, \tag{47}$$

$$\delta_E(\mathbf{r}-\mathbf{r}') = \sum_\epsilon B_\epsilon \frac{1}{g_p}\frac{\partial\psi_\epsilon^{(E)}}{\partial p}, \tag{48}$$

where g_p is the metric coefficient.

[26] For the waveguide problem, because $\frac{\partial\psi_\mu^{(H)}}{\partial p} \sim \psi_\mu^{(H)}$ and $\frac{\partial\psi_\epsilon^{(E)}}{\partial p} \sim \psi_\epsilon^{(E)}$, and because both $\psi_\mu^{(H)}$ and $\psi_\epsilon^{(E)}$ are complete sets, the expansions in (47) and (48) are complete. It should be point out that although the above expressions are derived in the neighborhood of the source point, they are valid for the whole waveguide region because the values of the complete expansions in (47) and (48) are zero outside the source point, which satisfies the properties of δ-function.

[27] For the magnetic-source magnetic-field DGFs for curved waveguides, $\hat{p} = \hat{\varphi}$, and the corresponding scalar eigenfunctions are

$$\psi_{mn}^{(H)} = B_{\mu_{mn}}(h_m\rho)\cos\frac{m\pi z}{c}e^{\mp j\mu_{mn}\varphi}, (m,n = 0,1,2,\ldots). \tag{49}$$

Using the orthogonal relation in Appendix A

$$\int_{\rho_1}^{\rho_2} \frac{1}{\rho}B_{\mu_{mn}}(h_m\rho)B_{\mu_{mn'}}(h_m\rho)d\rho = \delta_{nn'}N_{\mu_{mn}}, \tag{50}$$

we can show that

$$\hat{\varphi}\hat{\varphi}'\delta(\mathbf{r}-\mathbf{r}') = \hat{\varphi}\hat{\varphi}'\delta(\varphi-\varphi')\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{2}{1+\delta_{0m}} \cdot \frac{B_{\mu_{mn}}(h_m\rho)B_{\mu_{mn}}(h_m\rho')}{cN_{\mu_{mn}}\rho\rho'}C_zC_{z'}, \tag{51}$$

where $C_z = \cos(m\pi z/c)$ and $C_z{}' = \cos(m\pi z'/c)$. Similar to the expansion of the source-singularity term $\hat{z}\hat{z}'\delta(\mathbf{r}-\mathbf{r}')$ for straight waveguides, the above expression is valid when both the source point and field point approach the waveguide walls. Since the end conditions have no effects on the δ-function, the source-singularity terms for cavities take the same form as that of waveguides.

4.2. Forward Derivation: First Procedure for Reformulation

[28] In this subsection, starting from the common expressions of DGFs for curved waveguides and cavities in the previous section, we will derive their new forms suitable for numerical computation. We refer to this procedure as the forward derivation. (Another procedure, the backward derivation, first assumes the existence of this new form, as discussed in the next subsection.)

[29] As an example, consider the magnetic-source magnetic field dyadic Green's functions. When the source and field points approach the curved waveguide walls, i.e., $\rho = \rho' = R$, where $R = \rho_1$ or ρ_2, the source and field points are $\mathbf{r} = (R, \varphi, z)$ and $\mathbf{r}' = (R, \varphi', z')$, respectively. All ρ components of $\mathrm{M}_{\epsilon mn}(\mathbf{r})$ and $\mathbf{N}_{\mu mn}(\mathbf{r})$ vanish since $B'_{\mu mn}(h_m\rho_{1,2}) = C_{\epsilon mn}(h_m\rho_{1,2}) = 0$.

[30] The first term inside the summation of (40) contains only the $\hat{\varphi}\hat{\varphi}'$-component, i.e.,

$$\begin{aligned}\mathbf{M}^{\pm}_{\epsilon_{mn}}(\mathbf{r})\mathbf{M}^{\mp}_{\epsilon_{mn}}(\mathbf{r}') &= U(\varphi'-\varphi)\mathbf{M}^{+}_{\epsilon_{mn}}(\mathbf{r})\mathbf{M}^{-}_{\epsilon_{mn}}(\mathbf{r}') \\ &\quad + U(\varphi-\varphi')\mathbf{M}^{-}_{\epsilon_{mn}}(\mathbf{r})\mathbf{M}^{+}_{\epsilon_{mn}}(\mathbf{r}') \\ &= \hat{\varphi}\hat{\varphi}' h_m^2 C_{\epsilon_{mn}}(h_m\rho)C_{\epsilon_{mn}}(h_m\rho')C_z C_{z'} \\ &\quad \cdot \Big[U(\varphi'-\varphi)\Phi^{+}_{\epsilon_{mn}}(\varphi)\Phi^{-}_{\epsilon_{mn}}(\varphi') \\ &\quad + U(\varphi-\varphi')\Phi^{-}_{\epsilon_{mn}}(\varphi)\Phi^{+}_{\epsilon_{mn}}(\varphi')\Big] \\ &= \hat{\varphi}\hat{\varphi}' h_m^2[C_{\epsilon_{mn}}(h_m\rho)C_{\epsilon_{mn}}(h_m\rho')] \\ &\quad \cdot C_z C_{z'}\Phi^{+}_{\epsilon_{mn}}(\varphi_m)\Phi^{-}_{\epsilon_{mn}}(\varphi_M), \end{aligned} \tag{52}$$

where $U(x)$ is the unit step function, $\varphi_m = \min(\varphi, \varphi')$, and $\varphi_M = \max(\varphi, \varphi')$.

[31] Denoting $\bar{\mathbf{g}}^{(N)} = \frac{1}{h_m^2}\mathbf{N}^{\pm}_{\mu mn}(\mathbf{r})\mathbf{N}^{\mp}_{\mu mn}(\mathbf{r}')$ and $\Phi^{\pm}(\varphi) = \Phi^{\pm}_{\mu mn}(\varphi)$, we have

$$\begin{aligned}\overline{\mathbf{g}}^{(N)} = \frac{B_{mn}(\rho,\rho')}{h_m^2 k^2}\Bigg\{ & U(\varphi'-\varphi)\left(\hat{\varphi}\frac{\partial^2}{\rho\partial\varphi\partial z} + \hat{z}h_m^2\right) \\ & \cdot\left(\hat{\varphi}'\frac{\partial^2}{\rho'\partial\varphi'\partial z'} + \hat{z}'h_m^2\right)S_z S_{z'}\Phi^{+}(\varphi)\Phi^{-}(\varphi') \\ & + U(\varphi-\varphi')\left(\hat{\varphi}\frac{\partial^2}{\rho\partial\varphi\partial z} + \hat{z}h_m^2\right) \\ & \cdot\left(\hat{\varphi}'\frac{\partial^2}{\rho'\partial\varphi'\partial z'} + \hat{z}'h_m^2\right)S_z S_{z'}\Phi^{-}(\varphi)\Phi^{+}(\varphi')\Bigg\}, \end{aligned} \tag{53}$$

where $B_{mn}(\rho,\rho') = B_{\mu_{mn}}(h_m\rho)B_{\mu_{mn}}(h_m\rho')S_z = \sin(m\pi z/c)$ and $S_{z'} = \sin(m\pi z'/c)$.

[32] Now we consider the components in (53). First, the diagonal components are

$$\begin{aligned}\hat{z}\hat{z}' g^{(N)}_{zz} &= \hat{z}\hat{z}'\frac{1}{k^2}B_{mn}(\rho,\rho')h_m^2 S_z S_{z'}\Phi^{+}(\varphi_m)\Phi^{-}(\varphi_M) \\ &= \hat{z}\hat{z}' B_{mn}(\rho,\rho')S_z S_{z'}\Phi^{+}(\varphi_m)\Phi^{-}(\varphi_M) \\ &\quad - \frac{1}{k^2}\left(\hat{z}\frac{\partial}{\partial z}\right)\left(\hat{z}'\frac{\partial}{\partial z'}\right)B_{mn}(\rho,\rho') \\ &\quad \cdot C_z C_{z'}\Phi^{+}(\varphi_m)\Phi^{-}(\varphi_M), \end{aligned} \tag{54}$$

$$\begin{aligned}\hat{\varphi}\hat{\varphi}' g^{(N)}_{\varphi\varphi'} &= \hat{\varphi}\hat{\varphi}'\frac{B_{mn}(\rho,\rho')}{k^2 h_m^2}\left(\frac{m\pi}{c}\right)^2 C_z C_{z'} \\ &\quad \cdot\frac{\partial}{\rho\partial\varphi_m}\frac{\partial}{\rho'\partial\varphi_M}\Phi^{+}(\varphi_m)\Phi^{-}(\varphi_M) \\ &= \hat{\varphi}\hat{\varphi}'\left(\frac{1}{h_m^2} - \frac{1}{k^2}\right)C_z C_{z'} \\ &\quad \cdot\frac{\partial^2}{\rho\rho'\partial\varphi_m\partial\varphi_M}B_{mn}(\rho,\rho')\Phi^{+}(\varphi_m)\Phi^{-}(\varphi_M). \end{aligned} \tag{55}$$

Then, from the relations

$$\frac{\partial}{\partial z}S_z S_{z'} = -\frac{\partial}{\partial z'}C_z C_{z'}, \quad \frac{\partial}{\partial z'}S_z S_{z'} = -\frac{\partial}{\partial z}C_z C_{z'}, \tag{56}$$

we obtain the first off-diagonal component

$$\begin{aligned}\hat{\varphi}\hat{z}' g^{(N)}_{\varphi z'} &= -\hat{\varphi}\hat{z}'\frac{B_{mn}(\rho,\rho')}{k^2\rho}\frac{\partial}{\partial z'}C_z C_{z'} \\ &\quad \cdot\Big[U(\varphi'-\varphi)\frac{\partial\Phi^{+}(\varphi)}{\partial\varphi}\Phi^{-}(\varphi') \\ &\quad + U(\varphi'-\varphi)\frac{\partial\Phi^{-}(\varphi)}{\partial\varphi}\Phi^{+}(\varphi')\Big]. \end{aligned} \tag{57}$$

[33] Using the differential formula of the step function and the symmetry of δ-function, we finally simplify this to

$$\begin{aligned}\hat{\varphi}\hat{z}' g^{(N)}_{\varphi z'} &= -\frac{1}{k^2}B_{mn}(\rho,\rho') \\ &\quad \cdot\left(\hat{\varphi}\frac{\partial}{\rho\partial\varphi}\right)\left(\hat{z}'\frac{\partial}{\partial z'}\right)C_z C_{z'}\Phi^{+}(\varphi_m)\Phi^{-}(\varphi_M). \end{aligned} \tag{58}$$

[34] Similarly, the other off-diagonal component can be obtained as

$$\begin{aligned}\hat{z}\hat{\varphi}' g^{(N)}_{z\varphi'} &= -\frac{1}{k^2}B_{mn}(\rho,\rho') \\ &\quad \cdot\left(\hat{z}\frac{\partial}{\partial z}\right)\left(\hat{\varphi}'\frac{\partial}{\rho'\partial\varphi'}\right)C_z C_{z'}\Phi^{+}(\varphi_m)\Phi^{-}(\varphi_M). \end{aligned} \tag{59}$$

[35] Combining the above components, we arrive at the more compact dyadic expression

$$\begin{aligned}\overline{\mathbf{g}}^{(N)} &= \hat{z}\hat{z}' B_{mn}(\rho,\rho')S_z S_{z'}\Phi^{+}(\varphi_m)\Phi^{-}(\varphi_M) \\ &\quad + \Bigg\{\hat{\varphi}\hat{\varphi}'\Bigg[\frac{1}{h_m^2}\frac{\partial^2}{\rho\rho'\partial\varphi_m\partial\varphi'_M} \\ &\quad + \frac{1}{k^2}\left(\frac{\partial^2}{\rho\partial\varphi\rho'\partial\varphi'} - \frac{\partial^2}{\rho\rho'\partial\varphi_m\partial\varphi_M}\right)\Bigg] \\ &\quad - \frac{1}{k^2}\nabla_t\nabla'_t\Bigg\}B_{mn}(\rho,\rho')C_z C_{z'}\Phi^{+}(\varphi_m)\Phi^{-}(\varphi_M), \end{aligned} \tag{60}$$

where

$$\nabla_t = \hat{\varphi}\frac{\partial}{\rho\partial\varphi} + \hat{z}\frac{\partial}{\partial z}, \quad \nabla'_t = \hat{\varphi}'\frac{\partial}{\rho'\partial\varphi'} + \hat{z}'\frac{\partial}{\partial z'}. \tag{61}$$

[36] Furthermore, we can show that the second term in the above $\hat{\varphi}\hat{\varphi}'$ component can be reduced to

$$\begin{aligned}&\left(\frac{\partial^2}{\rho\partial\varphi\rho'\partial\varphi'} - \frac{\partial^2}{\rho\rho'\partial\varphi_m\partial\varphi_M}\right)\Phi^{+}_{\mu_{mn}}(\varphi_m)\Phi^{-}_{\mu_{mn}}(\varphi_M) \\ &= \delta(\varphi-\varphi')\Big[\Phi^{+'}(\phi')\Phi^{-}(\phi') - \Phi^{-'}(\phi')\Phi^{+}(\phi')\Big] \\ &= \delta(\varphi-\varphi')\Delta_{\mu_{mn}} \end{aligned} \tag{62}$$

where the following formula [*Jones*, 1982]

$$\alpha(x)\delta'(x-x_0) = \alpha(x_0)\delta'(x-x_0) - \alpha'(x_0)\delta(x-x_0), \tag{63}$$

has been used. Using the above formulas in the second term of the summation in (40), we then have

$$\frac{-k}{j\omega\mu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{mn}h_{mn}^2\hat{\varphi}\hat{\varphi}'\frac{1}{k}C_zC_{z'}\left(\frac{\partial^2}{\rho\partial\varphi\rho'\partial\varphi'} - \frac{\partial^2}{\rho\rho'\partial\varphi_m\partial\varphi_M}\right)B_{mn}(\rho,\rho')\Phi^+(\varphi_m)\Phi^-(\varphi_M) = \frac{\hat{\varphi}\hat{\varphi}'}{j\omega\mu}\delta(\varphi-\varphi')\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{2}{1+\delta_{0m}}\frac{B_{mn}(\rho,\rho')}{cN_{\mu_{mn}}\rho\rho'}C_zC_{z'}, \tag{64}$$

which is nothing but the negative magnetic source-singularity term, the last term of (40). Therefore, the expression of the magnetic-source magnetic-field DGF in (40) reduces to

$$\begin{aligned}\overline{\mathbf{G}}_H^{(m)}\big|_{\rho=\rho'=R} = {}& \frac{-k^2}{j\omega\mu}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{2}{(1+\delta_{0m})}\frac{\left[C'_{\epsilon_{mn}}(h_m\rho_2)\right]^2}{c\Delta_{\epsilon_{mn}}N_{\epsilon_{mn}}} \cdot \hat{\varphi}\hat{\varphi}'C_zC_{z'}\Phi^+_{\epsilon_{mn}}(\varphi_m)\Phi^-_{\epsilon_{mn}}(\varphi_M) \\ & - \frac{k^2}{j\omega\mu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{2}{(1+\delta_{0m})}\frac{\left[B_{\mu_{mn}}(h_m\rho_2)\right]^2}{c\Delta_{\mu_{mn}}N_{\mu_{mn}}} \cdot \left\{\hat{z}\hat{z}'S_zS_{z'} + \left[\hat{\varphi}\hat{\varphi}'\frac{1}{h_mR^2}\frac{\partial^2}{\partial\varphi_m\partial\varphi_M} - \frac{1}{k^2}\nabla_t\nabla_t'\right]C_zC_{z'}\right\}\Phi^+_{\mu_{mn}}(\varphi_m)\Phi^-_{\mu_{mn}}(\varphi_M).\end{aligned} \tag{65}$$

Note that in this new expression, the δ-function term disappears, and the operator $\nabla_t\,\nabla_t'$ is introduced. It is the operator $\nabla_t\,\nabla_t'$ that increases the order of singularity associated with the dyadic Green's functions. The introduction of $\nabla_t\nabla_t'$ is convenient for the follow-up numerical treatment when applying the moment method (MoM) to surface integral equations. Specifically, when evaluating the double surface integrals of the MoM self-coupling matrix elements, by using Gaussian theorem, we can transfer the above differential operators in the dyadic Green's function to the basis function and test function, respectively, so that the order of singularity in the integrand due to the dyadic Green's function will be reduced to first order from third order, greatly accelerating the convergence rate of the series solution.

4.3. Backward Derivation: A More General Reformulation Procedure

[37] In the previous subsection, starting with the common expression of DGFs, we derive a new form of the DGFs for curved waveguides and cavities on the curved walls. This form is also valid for an arbitrary waveguide whose walls are parallel to orthogonal curvilinear coordinate directions. Here we present a general, heuristic procedure for the reformulation of DGFs for such a waveguide.

[38] Taking curved waveguides as an example, let $c = a$, and $\rho = \rho' = \rho_2$. Equation (40) can then be written as

$$\begin{aligned}\overline{\mathbf{G}}_H^{(m)} = {}& \frac{1}{j\omega\mu}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\left[C'_{\epsilon_{mn}}(h_m\rho_2)\right]^2}{(1+\delta_{0m})N_{\epsilon_{mn}}aj\epsilon_{mn}}\hat{\varphi}\hat{\varphi}'C_zC_{z'} \cdot e^{-j\epsilon_{mn}|\varphi-\varphi'|} + \frac{1}{j\omega\mu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left[B_{\mu_{mn}}(h_m\rho_2)\right]^2}{(1+\delta_{0m})N_{\mu_{mn}}a} \\ & \cdot\left\{\frac{1}{\rho_2}\left[\pm\hat{\varphi}\hat{z}'C_zS_{z'}\mp\hat{z}\hat{\varphi}'S_zC_{z'}\right] + \frac{h_m^2}{j\mu_{mn}}\hat{z}\hat{z}'S_zS_{z'} - \frac{j\mu_{mn}}{h_{mn}^2\rho_2^2}\left(\frac{m\pi}{a}\right)^2\hat{\varphi}\hat{\varphi}'C_zC_{z'}\right\}e^{-j\mu_{mn}|\varphi-\varphi'|} \\ & - \frac{\hat{\varphi}\hat{\varphi}'}{j\omega\mu}\delta(\mathbf{r}-\mathbf{r}')\Big|_{\rho=\rho'=\rho_2},\end{aligned} \tag{66}$$

where the source-singularity term is expanded according to (47).

[39] Now we define a function $g_{mn}(\mathbf{r},\mathbf{r}')$ for the expansion function of the $\hat{\varphi}\hat{\varphi}'$ component within the second summation in (66) as follows,

$$g_{mn}(\mathbf{r},\mathbf{r}') = B_{\mu_{mn}}(h_m\rho)B_{\mu_{mn}}(h_m\rho')\,C_zC_{z'}e^{-j\mu_{mn}|\varphi-\varphi'|}. \tag{67}$$

Note that $g_{mn}(\mathbf{r},\mathbf{r}')$ corresponds to the expansion functions of $\hat{\varphi}\hat{\varphi}'$ component of $\mathbf{N}^{\pm}_{\mu mn}(\mathbf{r})$ $\mathbf{N}^{\pm}_{\mu mn}(\mathbf{r})$ wave function in (40), and its transverse functions take the same form as those of the expansion functions for δ_H $(\mathbf{r} - \mathbf{r}')$ function.

[40] Using the relations

$$\begin{aligned}\frac{\partial}{\partial\varphi}e^{-j\mu_{mn}|\varphi-\varphi'|} &= -\frac{\partial}{\partial\varphi'}e^{-j\mu_{mn}|\varphi-\varphi'|} \\ &= \pm j\mu_{mn}e^{-j\mu_{mn}|\varphi-\varphi'|} \quad \varphi\gtrless\varphi',\end{aligned} \tag{68}$$

$$\frac{\partial^2}{\partial\varphi\partial\varphi'}e^{-j\mu_{mn}|\varphi-\varphi'|} = \mu_{mn}^2e^{-j\mu_{mn}|\varphi-\varphi'|} + 2j\mu_{mn}\delta(\varphi-\varphi'), \tag{69}$$

and applying $\nabla_t\nabla_t'$ to (67), we arrive at

$$\begin{aligned}\nabla_t\nabla_t'\left[C_zC_{z'}e^{-j\mu_{mn}|\varphi-\varphi'|}\right] = {}& \hat{\varphi}\hat{\varphi}'\frac{2j\mu_{mn}}{\rho\rho'}\delta(\varphi-\varphi')C_zC_{z'} + \left\{\hat{\varphi}\hat{\varphi}'\frac{\mu_{mn}^2}{\rho\rho'}C_zC_{z'} + j\mu_{mn}\left(\frac{m\pi}{a}\right)^2 \cdot \left[\mp\hat{\varphi}\hat{z}'\frac{1}{\rho}C_zS_{z'} \pm \hat{z}\hat{\varphi}'\frac{1}{\rho'}S_zC_{z'}\right]\right.\end{aligned}$$

$$+ \hat{z}\hat{z}' \left(\frac{m\pi}{a}\right)^2 S_z S_{z'} \Bigg\} \cdot e^{-j\mu_{mn}|\varphi-\varphi'|} \quad \varphi \gtrless \varphi'. \tag{70}$$

[41] Substituting (70) into (66) and rearranging, we finally obtain

$$\begin{aligned}\bar{\mathbf{G}}_H^{(m)} = {} & \frac{k^2}{j\omega\mu} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\left[C'_{\epsilon_{mn}}(h_m\rho_2)\right]^2}{(1+\delta_{0m})N_{\epsilon_{mn}} a j \epsilon_{mn}} \\ & \cdot \hat{\varphi}\hat{\varphi}' C_z C_{z'} e^{-j\epsilon_{mn}|\varphi-\varphi'|} \\ & + \frac{k^2}{j\omega\mu} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\left[B_{\mu_{mn}}(h_m\rho_2)\right]^2}{(1+\delta_{0m})N_{\mu_{mn}} a j \mu_{mn}} \\ & \cdot \left\{ \left[\hat{z}\hat{z}' S_z S_{z'} + \hat{\varphi}\hat{\varphi}' \frac{\mu_{mn}^2}{h_m^2 \rho_2^2} C_z C_{z'}\right] e^{-j\mu_{mn}|\varphi-\varphi'|} \right. \\ & \left. - \frac{1}{k^2} \nabla_t \nabla'_t C_z C_{z'} e^{-j\mu_{mn}|\varphi-\varphi'|} \right\},\end{aligned} \tag{71}$$

which is the desired results. Again similar to (65), in the above expression the piecewise function only involves the difference $\varphi - \varphi'$ given in explicit form. In comparison with the expressions of the usual DGFs in piecewise function form, this new expression is more suitable for numerical treatment of the integration in the source region.

[42] The above heuristic procedure can be generalized to the reformulation of the DGFs for an arbitrary waveguide whose walls are parallel to orthogonal curvilinear coordinate directions (u, v, p). Without loss of generality, we define $\hat{p}$ as the propagating direction, and $u = u' = u_0$ (at the wall). We give a general reformulation procedure as follows: (1) Expand the $\delta(\mathbf{r} - \mathbf{r}')$ function term using (47) or (48), depending on whether the source is magnetic or electric. (2) Introduce the tangential derivatives at the coordinate plane coincident with the waveguide wall

$$\begin{aligned}\nabla_t &= \hat{p}\frac{1}{g_p}\frac{\partial}{\partial p} + \hat{v}\frac{1}{g_v}\frac{\partial}{\partial v}, \\ \nabla'_t &= \hat{p}'\frac{1}{g_{p'}}\frac{\partial}{\partial p'} + \hat{v}'\frac{1}{g_{v'}}\frac{\partial}{\partial v'},\end{aligned} \tag{72}$$

and evaluate the term $(-1/k^2)\nabla_t\nabla'_t\, g_{mn}(\mathbf{r} - \mathbf{r}')$, where $g_{mn}(\mathbf{r} - \mathbf{r}')$ is the expansion function of the $\hat{p}\hat{p}'$-component of $\mathbf{N}^{\pm}(\mathbf{r})\mathbf{N}^{\mp}(\mathbf{r}')$ function in the common DGF expression, whose transverse functions take the same form as those of the expansion function of the $\delta(\mathbf{r} - \mathbf{r}')$ function. (3) Substitute the above result into the original expression of the DGF, cancel the source-point term, and rearrange the terms in the expression into the desired form with the vector operator $(-1/k^2)\nabla_t\nabla'_t$. The procedure has been used for the transformation of the DGFs for straight waveguides with rectangular and sectoral cross sections.

[43] Finally, it is worth pointing out that when $\rho_2 \to \infty$ ($\rho_2 - \rho_1$ = constant), using the large argument asymptotic expansion of Bessel functions of both real and imaginary orders and modified Bessel functions of imaginary orders [*Fan and Yang*, 1994a, 1994b, 1995], we can show that the expressions of DGFs for curved waveguides and cavities derived above reduce to those of rectangular straight waveguides and cavities.

5. Conclusions

[44] The dyadic Green's functions for curved waveguides and cavities are derived through the Lorentz reciprocity theorem and the LSE and LSM mode orthogonality based on the concept of power flow, and by adding the source singularity terms. These DGFs are then reformulated into a new form suitable for numerical computation when both the source and field points are located at the same waveguide wall. A general reformulation procedure is proposed for DGFs for an arbitrary waveguide. The DGFs derived can be used for solving problems with curved waveguide coupling and radiating slots, as well as waveguide junction problems.

Appendix A: Orthogonality of the Radial Functions

[45] Let $Z_\alpha(h_m\rho) = B_{\mu mn}(h_m\rho)$ or $C_{\epsilon mn}(h_m\rho)$, and $Z_\beta(h_m\rho)$ $Z_\beta(h_m\rho) = B_{\mu mn'}(h_m\rho)$ or $C_{\epsilon mn'}(h_m\rho)$. Then $Z_\alpha(h_m\rho)$ and $Z_\beta(h_m\rho)$ satisfy the Bessel's equations

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dZ_\alpha(h_m\rho)}{d\rho}\right] + \left(h_m^2 - \frac{\alpha^2}{\rho^2}\right)Z_\alpha(h_m\rho) = 0, \tag{A1}$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dZ_\beta(h_m\rho)}{d\rho}\right] + \left(h_m^2 - \frac{\beta^2}{\rho^2}\right)Z_\beta(h_m\rho) = 0. \tag{A2}$$

Multiplying (A1) by $\rho Z_\beta(h_m\rho)$ and (A2) by $\rho Z_\alpha(h_m\rho)$, taking the differences, and integrating from ρ_1 to ρ_2, we obtain

$$\begin{aligned}\int_{\rho_1}^{\rho_2} \frac{1}{\rho} Z_\alpha(h_m\rho) Z_\beta(h_m\rho)\, d\rho = {} & \frac{1}{\alpha^2 - \beta^2} \\ & \cdot \Big\{ h_m\rho\big[Z'_\alpha(h_m\rho)Z_\beta(h_m\rho). \\ & - Z'_\beta(h_m\rho)Z_\alpha(h_m\rho)\big]\Big\}_{\rho_1}^{\rho_2}.\end{aligned} \tag{A3}$$

Since $B'_{\mu_{mn,n'}}(h_m\rho_{1,2}) = C_{\epsilon_{mn,n'}}(h_m\rho_{1,2}) = 0$, when $n \neq n'$ ($\mu_{mn} \neq \mu_{mn'}$ or $\epsilon_{mn} \neq \epsilon_{mn'}$), i.e., $\alpha \neq \beta$, the right-hand side of (A3) is equal to zero; while when $n = n'$ ($\mu_{mn} = \mu_{mn'}$ or $\epsilon_{mn} = \epsilon_{mn'}$), i.e., $\alpha = \beta$, the right-hand side is

indeterminate. An application of the L'Hospital rule results in

$$\int_{\rho_1}^{\rho_2} \frac{1}{\rho}[Z_\alpha(h_m\rho)]^2 d\rho = \frac{1}{2\alpha}\left\{(h_m\rho)\left[Z_\alpha(h_m\rho)\frac{\partial Z'_\alpha(h_m\rho)}{\partial\alpha} - Z'_\alpha(h_m\rho)\frac{\partial Z_\alpha(h_m\rho)}{\partial\alpha}\right]\right\}_{\rho_1}^{\rho_2}. \quad \text{(A4)}$$

Finally, we have

$$\int_{\rho_1}^{\rho_2} \frac{1}{\rho} B_{\mu_{mn}}(h_m\rho) B_{\mu_{mn'}}(h_m\rho) d\rho = \delta_{nn'} N_{\mu_{mn}}, \quad \text{(A5)}$$

and

$$\int_{\rho_1}^{\rho_2} \frac{1}{\rho} C_{\epsilon_{mn}}(h_m\rho) C_{\epsilon_{mn'}}(h_m\rho) d\rho = \delta_{nn'} N_{\epsilon_{mn}}, \quad \text{(A6)}$$

where the normalization constants, $N_{\mu_{mn}}$ and $N_{\epsilon_{mn}}$, are given in (15)–(18).

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