Absolute distribution, of Markov processes, 279
Absolute moments, 26
Absolute state probability of Markov processes, 279
with single-server Markovian queues, 289
Absorbing states, 275
busy period distribution and, 301
Abuse of differential equations, 124
with statistics, 54
Algebraic curves, in computing expected queue lengths and waiting times, 341
Almost sure events, 3
Alternative hypothesis ($H_1$), 75, 76
Analysis of variance (ANOVA), 76
Analytic functions, 195
Aperiodic Markov chains, 275, 276
Aperiodic states, 275
Approximation, in solving linear ODEs, 195
Arithmetic average, 17, 58–59
Arrivals
customer impatience and, 308–309
to $M/M/m$ queues with constant balking and exponential reneging, 332
to multiserver parallel systems, 312
priority of receiving, 329
to queues, 284–285
to single processor with splitting and delayed feedback, 334–335, 336
to single-server Markovian queues, 286, 288, 289–290, 303
Associated homogeneous equations, in method of undetermined coefficients, 178, 179, 180–181, 182, 183
Associativity, of convolution operator, 103
Auxiliary equations, 166
Average, 58–59
arithmetic, 17
weighted, 17
Average queue length, with single-server Markovian queues, 287–291, 291–294, 297
Axiom of choice, 21
Axioms of probability, 3–4
Back substitution, 155
Bacterial cultures, growth of, 161–162
Balance equations
for $M/M/1$ queues, 295, 299
for multiserver parallel systems, 320–323, 330–331
for single processor with splitting and delayed feedback, 337–341
for single-server Markovian queue with state-dependent balking, 309–311
Balked tasks (balking)
many-server queues with, 328–334
$M/M/1$ queue with state-dependent, 308–311
$M/M/m$ queues with constant, 332, 333–334
in queueing theory, 284
priority $M/M/2$ queues with constant, 328–332
Ball and urn problems, 6–7. See also Urn model
Batch service queue discipline, 285
Bayes’ formula, 5, 6, 7
Bayes’ theorem, $M/M/1/N$ queues and, 307
Bell-shaped curve, 33, 34
Bernoulli counting process, 270–271
Bernoulli equations, 156–158
Bernoulli processes, 270–271
Bernoulli random variables, 19
Bernoulli trials, 10–11, 12, 245–246
in Markov chains, 272–273
Bessel functions
busy period distribution and, 302
$M/M/1/N$ queueing system and, 305
$M/M/1$ transient queue length
distribution and, 291–294
Bessel’s equation, xii
Biased estimators, 68
Bilateral Z-transform, 105, 106–107, 108, 109–110
Bimodal distributions, 59
Binomial distribution, 19, 245–246, 313
Bernoulli counting process and, 270, 271
Binomial distribution function, defined, 11
Binomial probability distribution
functions, negative, 12–13
Binomial random variables, 11, 16
Binomial waiting time, 12–13
Biological map model, in modeling
population growth, 264
Birth and death (birth–death; B-D)
processes, 281–284
defined, 282–283
queueing systems as, 284–285
Borel sets, 21–22
defined, 22
Boundary conditions (BC)
in gambler’s ruin problem, 228–229
for PDEs, 210–211, 212–213
Boundary value problem, computing
expected queue lengths and waiting
times as, 342
Box plots, in statistics, 57
Brownian motion, 210
Buffers
in queueing theory, 285
for single processor with splitting and
delayed feedback, 334–335
Buffer size, for $M/M/1/N$ queueing
system, 303
Bulk service queue discipline, 285
Busy cycle, 301
Busy period distribution, for $M/M/1$
queues, 300–303
Busy periods, 300–303
in queueing theory, 285
Calculus. See also Derivatives;
Differential entries; Integral entries;
Partial derivatives
invention of, 121
this book and, xii
Calculus of differences, 218, 219
differential calculus vs., 219–220
Cartesian product, 41
Cauchy–Euler equations, 188–190
Cauchy probability distribution, 26
Cauchy–Riemann conditions, 206
Cauchy–Riemann equations, 206–208
Causality, 77
Causal models, 77
Centering, 18
Central limit theorem, 63, 64–67, 74
Central moments, 18
Central tendency, measures of, 58–59
Chance experiments, 2
Chaotic regimes, in modeling population
growth, 261, 262–263, 264
Chapman–Kolmogorov equations, 272,
277, 279–280. See also Kolmogorov
equations
for many-server parallel queues with
feedback, 326
M/M/1/N queueing system and, 304–305
for single-server Markovian queues, 288
for transition probability density function, 280–281
for types of Markov processes, 281
Characteristic equation method
for solving first-order linear nonhomogeneous difference equations, 231–237, 239–240, 242
for solving gambler’s ruin problem, 229
for solving nth-order linear homogeneous difference equations, 225–226, 230–231
Characteristic equations, 166, 192, 225–226, 227–228
partial differential equations and, 207
in solving Cauchy–Euler equations, 190
in solving linear homogeneous second-order differential equations, 169–175
in variation of parameters method, 187–188
Chebyshev’s inequality, 51–52
Chi-square ($\chi^2$) distribution, table of values for, 361–362
Chi-square ($\chi^2$) random variables, 32
Clairaut’s theorem, 123
Class boundaries, in statistics, 57
Classes, in statistics, 56–57
Class of events, 22
Closed queues, 284
Coefficient of correlation, 46–48
Coefficients, recursive formula for, 198.
See also Fourier coefficients; Linear constant-coefficient differential equations; Linear homogeneous difference equations with constant coefficients; Linear nonhomogeneous difference equations with constant coefficients; Linear second-order differential equations with constant coefficients; Method of undetermined coefficients; Regression coefficients
Coin flipping experiments, 53, 270
Combined waiting and loss system, in queueing theory, 285
Commutativity, of convolution operator, 103
Complementary equations, 176–177
Complements, of measurable sets, 21
Complex conjugate roots, in solving nth-order linear homogeneous difference equations, 225, 226–227
Complex functions, in computing expected queue lengths and waiting times, 341–350, 351, 352
Complex plane, $Z$-transforms and, 104, 105
Complex variables
Cauchy–Riemann equations and, 206
in computing expected queue lengths and waiting times, 341–350, 351, 352
probability generating functions and, 111–112
Computers, difference equations and, 219
Conditional probability, 4, 5
Conditional probability density function (pdf), 48
Conditional probability space, 4 with single-server Markovian queues, 289
Confidence interval, 72
constructing, 72–75
with MINITAB software, 61
standard normal, 74
Student’s $t$-distribution, 74–75
Conformal mapping, in computing expected queue lengths and waiting times, 341
Constant balking
M/M/m queues with, 332, 333–334
priority M/M/2 queues with, 328–332
Continuity, differential equations and, 124
Continuous functions, Laplace transforms of, 94–95
Continuous models, discrete models *vs.*, xii
Continuous probability density function (pdf), 24
cumulative distribution function and, 25
discrete probability density function *vs.*, 24
Continuous probability distribution functions, 26–41
Continuous random variables, 8, 20–25
conditional probability density function of, 48
defined, 24
joint bivariate probability density function of, 48
marginal probability density function of, 48
moments of, 25–26
Continuous random vectors, 48–49
Continuous sample spaces, 20
Continuous state, continuous space parameter Markov processes, 281, 282
Continuous state, discrete space parameter Markov processes, 281
Continuous-time Markov processes, 278–279
Continuous-time stochastic processes, 267
Convergence
  Laplace transforms and, 95
  stochastic, 50
Convergence in probability, 50
Convolution of two functions, 103
  Laplace transform of, 103–104
Convolution of two sequences, 114
Convolution operator, 103
Convolution theorem, 103–104
Cooling, Newton’s law of, 128, 162–163, 163–164
Correlation, 77
Cosine (cos), *See also* Trigonometric functions
  Fourier series and, 91–94
  Laplace transform of, 96–97
Countably additive axiom, 23
Counters, in queueing theory, 285
Counting processes, 268, 270–271, 271–272
Covariance (Cov), 45–46
  properties of, 46
Cramer’s rule, in variation of parameters method, 186, 187
Critically damped spring–mass system, 201, 202–203
Critical region, 75
Critical value, 75
Cumulative distribution function (cdf), 25
defined, 10
  likelihood function and, 68–69
  limiting, 49–50
  of lognormal random variable, 35
Curl, xii
Customer impatience, in queueing theory, 308–311
Customers, in queues, 284
Damped-free vibration, 200–203
Damping, 199–200, 200–203
Damping factors, 201
Data displays, in statistics, 57
Data points
  dispersion of, 59–61
  mean, median, and mode of, 58–59
  in statistics, 53–54, 55–58
Data sets
  dispersion of, 59–61
  skewness of, 59
  in statistics, 53
Decay models. *See Growth/decay models*
Deciles, 59
Degree of confidence, 72
Delayed feedback, single-server
  Markovian queueing system with, 334–352
Delayed renewal processes, 268
Delay stations, for single processor with splitting and delayed feedback, 335, 336
Delay time, in solving the logistic differential equation, 138–140
Delta functions, 100–101, 254, 280–281, 304, 308, 321, 326
Density dependent equation, in modeling population growth, 263
Density-dependent regulation, in modeling population growth, 264
Density functions, 24
   Cauchy probability distribution with, 26
   exponential, 28–29, 30
   rectangular, 27
   uniform, 27
Denumerable sets, stochastic processes and, 267
Dependent events, 44
   Markov processes and, 269
Derivative operator \((D)\), 190, 191, 192
Derivatives
   in differential equations, 123, 124–127
   difference operators vs., 219–220
   Laplace transform of, 99–100
Descriptive statistics, 1
Deterministic arrivals, in queueing theory, 284
Deviation, 18
Difference equations, 218–266. See also
   Differential–difference equations
   applications of, 218–219
   basic terminology for, 220–224
   defined, 221
   differential equations vs., 219
   first-order linear nonhomogeneous with constant coefficients, 231–240
   history of, 218
   linear, 223–224
   linear homogeneous with constant coefficients, 224–231
   nonlinear, 218, 223–224, 259–264
   population growth/decay models and, 218
   probability generating functions in solving, 111
   real-world problems and, 218
   second-order linear nonhomogeneous with constant coefficients, 240–242
   solving differential equations via, 242–243
   solving via generating functions method, 245–252
   this book and, xi, xii
Difference operators, derivatives vs., 219–220
Differentiable functions, 127, 129
Differential, 121, 130
Differential calculus
   calculus of differences vs., 219–220
   invention of, 121
Differential–difference equations, 219, 253–259. See also Difference equations
   busy period distribution and, 301–303
   for \(M/M/1/N\) queueing system, 304
   for \(M/M/m\) queues with constant balking and exponential reneging, 333–334
   for multiserver parallel systems, 313–314
   for single processor with splitting and delayed feedback, 336–341, 341–349, 352
   for single-server Markovian queue with feedback, 307–308
   for single-server Markovian queue with state-dependent balking, 309–311
Differential–difference equation systems, for modeling Poisson processes, 254–259
Differential equations, 121–217. See also
   Ordinary differential equations (ODEs); Partial differential equations (PDEs)
   basic concepts and definitions for, 121–130
   defined, 123–127
   difference equations vs., 219
   exact, 144–153
   exactness test for, 147–148
   existence and uniqueness of solutions of, 130–132, 167
   history of, 121
   integrating factors of, 151, 152–153
   linear, 140–144
   not exact, 151–153
   operators in solving systems of, 191–193
   separable, 132–140
Differential equations (cont’d)
solving via elimination method, 191–193
this book and, xi, xii
Differential of \( z \), 145–146
Differential operators, 190–191
in solving differential equations, 191–193
Dirac delta function (\( \delta \)), 100–101
probability density function as, 280–281
Dirichlet-type boundary conditions (BC), 210, 211, 212
Discrete distributions, properties of, 16–20
Discrete functions, Laplace transforms of, 95
Discrete models, continuous models vs, xii
Discrete probability density function (pdf)
continuous probability density function vs., 24
cumulative distribution function and, 25
defined, 9–10
Discrete random variables, 7–16
defined, 8
as functions, 8
functions of, 49–50
Markov’s inequality and, 50–51
moment generating functions of, 114–115
moment of order \( r \) and, 17–18
moments of, 16–20
probability distributions of, 9
Discrete random vector, 41–42, 43
Discrete-space Markov processes, 278
Discrete state, continuous space parameter Markov processes, 281–284
Discrete state, discrete space parameter Markov processes, 281
Discrete state spaces, discrete-time stochastic processes with, 269
Discrete-time Laplace transform, \( Z \)-transform as, 104–105
Discrete-time Markov chain, 269
Discrete-time stochastic processes, 267
with discrete state spaces, 269
Discretization, differential equations and, 124
Dispersion, measures of, 59–61
Distinct real roots, in solving \( n \)th-order linear homogeneous difference equations, 225, 230
Distribution functions
exponential, 28–29, 30
gamma, 30
generating functions method and, 248
joint probability mass functions and, 42–44
limiting, 49–50
rectangular, 27
uniform, 26–28
Distribution of a population, 58
Distribution of queue length, for \( M/M/m \) queues with constant balking and exponential reneging, 333–334
Distributions
central limit theorem and, 64–67
mean, median, and mode of, 58–59
Distributivity, of convolution operator, 103
Divergence, xii
Domain
of complex variable functions, 206
of partial differential equations, 204–205
Domain of a function, 122
Dot plots, in statistics, 57
Double exponential distribution, 40–41
Double real roots, in solving \( n \)th-order linear homogeneous difference equations, 225, 226, 230–231
Doubly stochastic matrix, for Markov chains, 269–270
Driving parameter, in modeling population growth, 264
Effective arrival rate, for \( M/M/1/N \) queueing system, 306
Effective service rate, 307
Eigenvalue method, \( M/M/1/N \) queueing system and, 304–305
Elasticity, Hooke’s law of, 128
Electromagnetic phenomena, partial differential equations for modeling, 204
Elementary events, 2

*Elements of Queueing Theory with Application* (Saaty), 284
Elimination method, 190–193, 340
Elliptic equations, 208, 209
Empty set (φ), 21
Epochs, in stochastic processes, 267–268
Equilibrium, in population growth models, 136–137
Equilibrium points, in modeling population growth, 260–261
Equiprobable sample spaces, 3
Ergotic Markov chains, 275, 276–277, 278
Ergotic states, 275
Erlang distribution, for stationary waiting time of a task, 323
Erlang distribution of order \( k (E_k) \), 30–32
Erlang law, 284
Erlang’s delay (second) formula, 321
Error bar, 76
Errors, in hypothesis testing, 75–76

Estimates, 68
Estimation, 68, 81. See also Interval estimation; Point estimation
Euclidean spaces (\( \mathbb{R}^n \))
    discrete random vectors and, 41
    Lebesgue measure on, 23
Euler, Leonhard, 90–91
Euler method, in solving differential equations, 242–243
Euler’s formula, 96–97, 108–109
Euler’s rule, 174
Even functions, 92–93
Even number factorial, 198
Event occurrence, 2
Events, 2. See also Dependent events;
    Independent events
    conditional probability of, 4, 5
    families of, 22
    Markov processes and, 268–269
mutually exclusive, 4
probabilities of, 3, 4
in stochastic processes, 267–268
Event times, in stochastic processes, 267–268
Exact differential, 146
Exact differential equations, 144–153
algorithm for solving, 148–150
defined, 146
exactness test and, 147–148
integrating factors of, 151, 152–153
Exactness test, 147–148
Existence of solutions
    of difference equations, 224, 238–239, 240
    of differential equations, 130–132, 167, 175–176
Exiting, in queueing theory, 285
Expectation, 16–17, 25
Expected queue length, for \( M/M/1/N \) queueing system, 306
Expected values, 16–17, 25
    of indicator functions, 18–19
Experiments, 2
Explicit solution, of a differential equation, 124–126, 127
Exponential density functions, 28–29, 30
Exponential distribution
    double, 40–41
    with single-server Markovian queues, 286, 287, 288
Exponential distribution functions, 28–29, 30
Exponential random variables, negative, 28–29, 30
Exponential reneging
    \( M/M/m \) queues with, 332, 333–334
    priority \( M/M/2 \) queues with, 328–332
Exponential service distribution, for
    single processor with splitting and delayed feedback, 335
Extreme data points, in statistics, 57
Extreme value distribution, 40–41
Factorial function (!), 31, 198
    generalized, 31
Factorial moments, probability generating function and, 113
Failed state, 78
Failure, in Bernoulli trials, 10, 11, 12
Failure rate function, 80
Failure—repair model with one machine and one repairman [FR(1,1)], 282
Families, of subsets, 22
Feedback, 307. See also Many-server parallel queues with feedback; M/M/1 queue with feedback; Parallel queues with feedback; Single processor with splitting and delayed feedback; Single-server Markovian queue with feedback in queueing theory, 285
queueing theory for many-server parallel queues with, 326–328
Fibonacci difference equation, 226
Finite buffers, in queueing theory, 285
Finite-buffer single-server Markovian (M/M/1/N) queues, queueing theory for, 303–307
Finite populations, random sampling from, 58
Finite sequences, defined, 220–221
Finite sets, stochastic processes and, 267
Finite-state (finite) Markov chain, 269, 275–276, 277
First come, first served (FCFS) queue discipline, 285
First in, first out (FIFO) queue discipline, 285
with multiserver parallel systems, 312, 329, 332
First moment, 16–17, 25
First-order derivatives, Laplace transforms of, 99–100
First-order difference equations generating functions method and, 247 homogeneous, 250 linear nonhomogeneous with constant coefficients, 231–240 solving systems of, 251–252
First-order differential equations, 124, 128 applications of, 159–164 exact, 144–145, 146 general form of, 130 linear, 140–144 separable, 132–133 solving by substitution method, 153–159
First-order partial differential equations, 203, 204, 205–208 applications of, 205–206
First quartile, 58
First shifting, of Laplace transform, 99
Fisher, R. A., 77
Force of mortality, 80
Forgetfulness property, of Markov chains, 270, 279
Forward Kolmogorov equations, 291
Fourier coefficients, 91–92, 92–93
Fourier cosine series, 93, 211
Fourier series, 90–94, 210, 211 defined, 91 partial differential equations and, 210–213
Fourier sine series, 93, 211
Fourier transforms (FT), xi, 90–94 classical, xii defined, 93–94 Laplace transforms vs., 94 Z-transforms vs., 104–105 Fréchet-type distribution, 40–41 Free undamped motion, 199–200 Frequency distributions, in statistics, 53–54, 55–58 Frequency distribution with classes, in statistics, 56–57 Functioning state, 78 Functions. See also Generating functions (G); Indicator functions; Point functions analytic, 195 convolution of, 103–104 differential equations and, 121, 122–130 differentials of, 130 discrete random variables as, 8 integrals of, 25 operators on, 190–191 partial differences of, 244 random variables as, 8 transforms of, 90–120 Functions of one variable, differential equations and, 121, 122 Functions of random variables, 49–53
Functions of two variables
differential equations and, 121, 122
partial derivatives and, 122, 123
Fundamental events, 2

Galton random variable, 35
Gambler’s ruin problem, 228–229
Gamma distribution function, 30
Gamma function (Γ), 30, 31–32
incomplete, 31
Gamma random variables, 29–30
Gaussian distribution, central limit
theorem and, 64–67
Gaussian random variables, 32–33
Generalized extreme value distribution,
41
Generalized factorial function, 31
Generalized hypergeometric function,
$M/M/m$ transient queue length
distribution and, 317
General logistic cumulative probability
density function, 35–36, 37–38
General logistic cumulative probability
distribution, 35–36, 37, 38
General solutions
of complementary equations, 176, 177
of difference equations, 222, 232–233,
233–234, 235, 236–237, 243
in method of undetermined
coefficients, 181, 182, 184
Generating function(s) method
$M/M/m$ transient queue length
distribution and, 315–320
with single-server Markovian queue
with state-dependent balking, 311
in solving difference equations,
245–252
in solving differential–difference
equations, 253–254
in solving differential–difference
equation systems, 256–259
Generating functions ($G$), 90, 111–116,
244
defined, 111–112
products of, 114
for single processor with splitting and
delayed feedback, 337–341
of sum of $n$ independent variables, 114
this book and, xii
Geometric probability distribution
function, 13
Geometric progression, 245
Good estimators, 68
Gradient, xii, 210
Gradient vector, 210
Graphic data displays, in statistics, 57
Grouping, in statistics, 53–54
Growth/decay models
first-order ODEs as, 159–162
for populations, 129–130, 135–140, 218
Gumbel-type distribution, 40–41
Haghighi, Aliakbar M., xiii
Harmonic functions, 209
Heat equations, 204, 208, 209
one-dimensional, 210–211, 212
three-dimensional, 210
Heat phenomena, 209
partial differential equations for
modeling, 203–204
Heaviside, Oliver, 100
Heaviside step function ($H$), 100–101,
106, 107
High-order derivatives, Laplace
transforms of, 100
Hilbert boundary value problem,
computing expected queue lengths
and waiting times as, 342
Histograms
with MINITAB software, 61
in statistics, 57
Homogeneous difference equations
first-order, 250
linear with constant coefficients,
224–231
Homogeneous Hilbert boundary value
problem, computing expected queue
lengths and waiting times as, 342
Homogeneous linear differential
equations, 140, 143, 153–154,
155–156, 160, 161, 223
with constant coefficients, 224–231
initial value problems for second-
order, 167
second-order, 164–175
solving second-order, 165–175
in variation of parameters method,
184–186
Homogeneous Markov chains, 270
Homogeneous Markov processes, 280
for many-server parallel queues with feedback, 326
Homogeneous partial differential equations, 204
Homogeneous transition probability density function (tpdf), 280–281
Hooke, Robert, 128
Hooke’s law of elasticity, 128
Hutchinson, G. E., 139
Hyperbolic equations, 208, 209
Hypergeometric function, \( M/M/m \) transient queue length distribution and, 317
Hypergeometric probability mass function (pmf), 14–15
Hypothesis testing, 67–68, 75–78
Idle period, 301
Image function (image), of a Laplace transform, 94–95
Impatient tasks, in queueing theory, 308
Implicit function theorem, 127
Implicit solution, of a differential equation, 126–127, 145, 146
Impossible events, 3
Incomplete gamma function, 31
Increments, for random counting processes, 268. See also Independent increments
Independent events, 4–5, 6, 7
discrete random vectors and, 42
Markov processes and, 268–269
Independent increments, in stochastic processes, 278
Independent random variables, 48–49
first moment and, 17
generating function of sums of, 114
Index sets \( (T) \), for stochastic processes, 267
Indicator functions, 9
expected values of, 18–19
first moment and, 17
Indicator random variables, 9
Inference, 67
Inferential statistics, 1, 67–75
Infinite buffers, in queueing theory, 285
Infinite populations, random sampling from, 58
Infinite sequences, defined, 220–221
Infinite-server queues, 285
Inhomogeneous linear difference equations, 223
Inhomogeneous partial differential equations (PDEs), 204
Initial conditions, 131
generating functions method and, 247, 249
for homogeneous second-order differential equations, 167, 168, 170
for partial differential equations, 210–211
for Poisson process, 254
Initial probability distributions, of Markov processes, 279
Initial state, in stochastic processes, 267
Initial value problems (IVPs), 131
analytic functions and, 196
for difference equations, 238, 239, 240–241, 243
for homogeneous second-order differential equations, 167, 168
Laplace transforms in solving, 194
for nonhomogeneous linear second-order differential equations, 175–176
Initial values, for difference equations, 223–224, 237–238
In-server queueing systems, 312
Instantaneous rate of change, 128
Instructors, this book and, xi
Integers, sequences and nonnegative, 220–221
Integral calculus, invention of, 121
Integral curve, 125
Integral–difference equations, 253
Integrals, 25
Laplace transforms of, 100
Integral sign \( (\int) \), 121
Integral theorems, xii
Integral transforms, 93–94
  Integrating factors, 151, 152–153
    in solving exact differential equations, 144–145
    in solving linear differential equations, 141–142, 143, 144
  Interarrival times
    for $M/M/1/N$ queueing system, 306
    with single-server Markovian queues, 286
  Intersections, 5
    of measurable sets, 21
  Interval estimation, 68, 72–75
  Interval estimator, 72
  Intervals, Lebesgue measure on, 23
  Inverse, of logistic pdfs and cdfs, 37–38
  Inverse Fourier transform, 94
  Inverse Laplace transform, 94–95, 98, 102–103
  Inverse transform of $Z$-transform, 105–106, 109–110
  Inverse Weibull probability distribution, 39
  Irreducible Markov chains, 275–276
  Items, in queues, 284
  Iterative method, in finding stationary
    waiting time of a task, 325
  Jobs, in queues, 284
  Jockeying, 330
    in queueing theory, 285, 308
  Joint bivariate probability density
    function (pdf), 48
  Joint parametric probability density
    function (pdf), 49
  Joint probability distribution, for single
    processor with splitting and delayed feedback, 341
  Joint probability mass functions (joint
    pmfs), 42–44
  Jury, E. L., 104
  KdV equation, 204
  Kernel function (kernel), 93–94
  Khintchine’s method, for stationary
    waiting time of a task, 323
  Known variance, 72–74
  Kolmogorov, Andrey, 21
  Kolmogorov axiomatization of
    probability, 21
  Kolmogorov equations, forward, 291. See
    also Chapman–Kolmogorov
    equations
  $k$-out-of-$n$ structures, 79
  Kronecker delta ($\delta$) function, 101, 254,
    304, 308, 321, 326
  $k$th moments, 26
  Kurtosis, 59
  Lagrange, Joseph-Louis, 91
  Laplace equation, 204, 208, 209
  Laplace, Pierre-Simon, Marquis de, 94
  Laplace transforms, xi, 90, 94–104
    busy period distribution and, 301–303
    classical, xii
    of convolution of two functions, 103–104
    defined, 94–95
    of derivatives, 99–100
    existence of, 98–99
    first shifting ($s$-shifting) of, 99
    Fourier transforms vs., 94
    of integrals, 100
    linearity of, 95–98
    multiserver parallel systems and, 313–314, 314–315, 316–317
    properties of, 95–104
    second shifting ($t$-shifting) of, 100–103
    with single-server Markovian queues, 291, 292, 293, 294
    in solving differential–difference
      equation systems, 255–256, 257–258
    in solving ODEs, 193–194
    tables of, 96, 97
    $Z$-transforms as discrete-time, 104–105
  Large numbers, laws of, 50, 52–53
  Last come, first served (LCFS) queue
    discipline, 285
  Laurent series, 90
  Law of total probability, 5, 45
  Laws of large numbers
    strong, 52–53
    weak, 50, 52, 53
  Lebesgue, Henri, 25
  Lebesgue integral, 21
  Kolmogorov axiomatization of
    probability, 21
  Kolmogorov equations, forward, 291. See
    also Chapman–Kolmogorov
    equations
  $k$-out-of-$n$ structures, 79
  Kronecker delta ($\delta$) function, 101, 254,
    304, 308, 321, 326
  $k$th moments, 26
  Kurtosis, 59
  Lagrange, Joseph-Louis, 91
  Laplace equation, 204, 208, 209
  Laplace, Pierre-Simon, Marquis de, 94
  Laplace transforms, xi, 90, 94–104
    busy period distribution and, 301–303
    classical, xii
    of convolution of two functions, 103–104
    defined, 94–95
    of derivatives, 99–100
    existence of, 98–99
    first shifting ($s$-shifting) of, 99
    Fourier transforms vs., 94
    of integrals, 100
    linearity of, 95–98
    multiserver parallel systems and, 313–314, 314–315, 316–317
    properties of, 95–104
    second shifting ($t$-shifting) of, 100–103
    with single-server Markovian queues, 291, 292, 293, 294
    in solving differential–difference
      equation systems, 255–256, 257–258
    in solving ODEs, 193–194
    tables of, 96, 97
    $Z$-transforms as discrete-time, 104–105
  Large numbers, laws of, 50, 52–53
  Last come, first served (LCFS) queue
    discipline, 285
  Laurent series, 90
  Law of total probability, 5, 45
  Laws of large numbers
    strong, 52–53
    weak, 50, 52, 53
  Lebesgue, Henri, 25
  Lebesgue integral, 21
Lebesgue measure, 23
Lebesgue–Stieltjes integral, 25
Left skew, 59
Leibniz, Gottfried Wilhelm, 121
Leibniz differentiation theorem, $M/M/m$
 transient queue length distribution and, 318–320
Length of a class, in statistics, 57
Level of confidence, 72
Level of significance of a test, 76
l’Hôpital’s rule, 115, 346, 349
Likelihood equations, 71
Likelihood function, 68–69, 70–72
Limiting distribution functions, 49–50
Limiting probability distribution of queue lengths, for $M/M/1/N$
 queueing system, 305–306
Linear algebra, in solving partial differential equations, 205
Linear Bernoulli equations, 156–157, 157–158
Linear combinations, 166
Linear constant-coefficient differential equations, 121
Linear constant-coefficient difference equations, 224–231, 231–243
Linear difference equations, 223–224
 first-order nonhomogeneous with constant coefficients, 231–243
 homogeneous, 223
 homogeneous with constant coefficients, 224–231
 linear differential equations vs., 229–230, 230–231
 nonhomogeneous (inhomogeneous), 223
 second-order nonhomogeneous with constant coefficients, 240–242
 systems of, 244–252
Linear differential equations, 140–144
Cauchy–Euler, 188–190
homogeneous, 140, 143, 153–154, 155–156, 160, 161
Laplace transforms in solving, 193–194
linear difference equations vs., 229–230, 230–231
nonhomogeneous, 140, 141–144, 154, 165
nonhomogeneous second-order with constant coefficients, 175–188
power series in solving, 195–199
solving homogeneous second-order, 165–175
Linear homogeneous difference equations with constant coefficients, 224–231
Linearity, of Laplace transforms, 95–98
Linearity condition, 167–168
Linearly dependent solutions, for homogeneous second-order differential equations, 167–168
Linearly independent solutions of complementary equations, 176
of homogeneous second-order differential equations, 167, 168–169, 171
in variation of parameters method, 184–186, 187–188
Linear nonhomogeneous difference equations with constant coefficients
first-order, 231–240
second-order, 240–242
Linear partial differential equations (PDEs), 203–204
Linear regression, 77
Linear second-order difference equations, 226–227
Linear second-order differential equations with constant coefficients, 165, 175–188
Little’s formula
 in finding stationary waiting time of a task, 324
$M/M/1/N$ queues and, 307
$M/M/1$ queues and, 300
Location parameter
 of logistic distributions, 36–37
of Weibull cumulative probability distribution function, 38
Logistic difference equation, in modeling population growth, 263–264
Logistic differential equations, 129–130
in modeling population growth, 260–261, 262
solving, 135–140
Logistic distributions, 35–38
  normal distributions vs., 36
Log likelihood function, 69, 70
Lognormal random variable, 35
Loss systems, 330
  in queueing theory, 285
Maclaurin series expansion, 252, 337, 341
Main service station, for single processor
  with splitting and delayed feedback, 335
Malthus, Thomas, 129
Malthusian growth model, 129, 160–161
Many-server parallel queues with
  feedback
  queueing theory for, 326–328
  stationary waiting time of a task for, 327–329
Many-server queueing systems, 312
Many-server queues with balking and
  reneging, queueing theory for, 328–334
Mappings, random variables as, 8
Marginal distributions, 47
Marginal probability density function
  (pdf), 48
Marginal probability mass functions
  (marginal pmfs), 45
Marked point processes, 268
Markov chains, 268–281
  Bernoulli trials in, 272–273
  discrete-time, 269
  finite-state/finite, 269, 275–276, 277
  forgetfulness property of, 270
  homogeneous, 270
  Polya’s urn model and, 274
  in solving differential–difference
  equation systems, 257
  states of, 275–276, 277
  stationary (steady-state) distributions
  and vectors of, 277–278
Markov counting process, 271–272
Markovian queueing systems, single-
  server with splitting and feedback, 334–352
Markovian queues, single-server,
  286–303
Markov processes
  absolute distribution of, 279
  birth and death, 281–284
  Chapman–Kolmogorov equations for,
    279–280
  discrete-space and continuous-time,
    278–279
  initial probability distributions of, 279
  for many-server parallel queues with
    feedback, 326
  transition probabilities of, 279
  transition probability density functions
    of, 280–281
Markov property, 269, 278, 279, 279
Markov’s inequality, 50–51
Mass transport, 209
Mathematical expectation, 16–17, 25
Mathematical induction, in solving
  first-order linear nonhomogeneous
  difference equations, 238
Mathematical models, real-life, 127–130
“Mathematics for engineers” textbooks, xi
MATLAB software, 72
  in solving differential equations, 243
  in solving the logistic differential
  equation, 137
Matrices. See also Stochastic matrices (P);
  Transition probability matrices (P)
  in Chapman–Kolmogorov equation,
    272, 279–280
  notation for, xii
stochastic, 269–270
Maximum, of a data set, 53
Maximum likelihood estimation (MLE), 68
  estimating Poisson parameter via,
    69–70
  estimating Weibull parameter via,
    70–72
Maximum likelihood estimator (MLE), 69
Maximum likelihood function, 68
Maxwell’s equations, xii
May, Robert M., 260, 264
Mean, 16–17, 25, 61–62, 62–63. See also
  Sample mean
central limit theorem and, 64–67
generating functions method and, 248
Mean (cont’d)
of logistic distributions, 36
of lognormal random variable, 35
with MINITAB software, 60–61
moment generating functions and,
115–116
of normal random variables, 33
of a Poisson random variable, 19–20
of a random variable, 19
with single-server Markovian queue
with state-dependent balking, 311
of a Weibull random variable, 39
Mean functioning time, 80
Mean sojourn time, for many-server
parallel queues with feedback, 328
Mean square, 60
Mean square error (MSE), of good
estimators, 68
Mean waiting time of a task, for
many-server parallel queues with
feedback, 328
Mean waiting times in line, 324
Measurable sets, 21
Measurable spaces, 22
Measurable subsets, 21
Measure (μ), 21. See also Probability
measure
Measures of central tendency, 58–59
Measures of dispersion, 59–61
Median, 58, 59
with MINITAB software, 60–61
Method of characteristics, partial
differential equations and, 207
Method of moments, in point estimation,
68
Method of substitution. See Substitution
method
Method of undetermined coefficients
for nonhomogeneous differential
equations, 178–184
in solving Cauchy–Euler equations, 190
Method of variation of parameters,
171
for nonhomogeneous differential
equations, 184–188
Middlemost observed value, 59
Military, balking and reneging and, 309
Minimum, of a data set, 53
Minimum variance estimators, 68
MINITAB software, 60–61
Mishev, Dimitar P., xiii
Misuse, in statistics, 54, 55–58
Mixed boundary conditions, 211
M/M/1/N queueing system, 303–307
M/M/1 queueing system, 286–303
M/M/1 queue with feedback, queueing
theory for, 307–308
M/M/1 queue with state-dependent
balking, queueing theory for,
308–311
M/M/2 loss system, 330
M/M/2 queues, with constant balking
and exponential reneging, 328–332
M/M/m queues
with constant balking and exponential
reneging, 332, 333–334
queueing theory for, 311–325
stationary queue length distribution
for, 320–323
stationary waiting time of a task with,
323–325
transient queue length distribution for,
312–320
Mode, 58, 59
Models, of population growth, 260–264.
See also Queueing models
Modified Bessel functions, busy period
distribution and, 302
Moment generating functions, 114–115
Moment of order r, 17–18
Moments. See also Method of moments
central limit theorem and, 64–67
of a continuous random variable,
25–26
of a discrete random variable, 16–20
generating functions method and, 248
Monotone functions, 21
Mortality, force of, 80
Multichannel queueing systems, 312
Multichannel servers, in queueing theory,
285
Multimodal distributions, 59
Multinomial probability mass functions
(multinomial pmfs), 45
Multiple real roots, in solving nth-order
linear homogeneous difference
equations, 230
Multiple-server queues, with balking and reneging, 328–334
Multiplicative law, 4
Multiprocessor systems, 312
Multiserver parallel queues
queueing theory for, 311–325
structure of, 312
Multiserver parallel systems, stationary waiting time of a task with, 323–325
Multiservers, in queueing theory, 285
Mutually exclusive events, 2, 4
Mutually independent random variables, 48–49
Negative binomial probability distribution functions, 12–13
Negative exponential random variable, 28–29, 30
Negative skew, 59
Neumann-type boundary conditions, 210–211, 212–213
Newton, Isaac, 121
Newton’s iterative method, 71–72
Newton’s law of cooling, 128, 162–163, 163–164
Newton’s method, 345, 346, 347, 349
Newton’s second law of motion, 128, 164–165, 200
Nondenumerable sets, stochastic processes and, 267
Nonfunctioning state, 78
Nonhomogeneous difference equations
first-order linear with constant coefficients, 231–243
second-order linear with constant coefficients, 240–242
Nonhomogeneous linear difference equations, 223
Nonhomogeneous linear differential equations, 140, 141–144, 154, 165
$M/M/m$ transient queue length distribution and, 315–316
operators in solving systems of, 191
second-order with constant coefficients, 175–188
solving, 141–144
Nonhomogeneous partial differential equations (PDEs), 204
Nonlinear difference equations, 218, 223–224, 259–264
history and applications of, 218–219
in modeling population growth, 260–264
Nonlinear differential equations, solving, 133–134
Nonlinear partial differential equations (PDEs), 204
Nonmeasurable sets, 21
Nonnegative integers, sequences and, 220–221
Nonnegative random variables, Chebyshev’s inequality and, 51–52
Normal cumulative distribution function (cdf), 34
Normal density curve, standard, 73
Normal distributions central limit theorem and, 64–67
logistic distribution vs., 36
standard, 73
Normalizing equations, 245
generating functions method and, 246–247
for Poisson process, 254
Normal probability density function (pdf), 33–34
Normal probability distribution, table of values for, 363–364
Normal random variables, 32–33
Notation in queueing theory, 285–286
with single-server Markovian queues, 288
used in this book, xii
$n$-steps transition probability, 271
$n$th factorial moment, probability generating function and, 113
$n$th-order linear homogeneous difference equations, with constant coefficients, 224–226
$n$th-step transition matrix, 272
Null events, 3
Null hypothesis ($H_0$), 75–76
Numerical solutions in solving differential equations, 242–243
for standard logistic differential equation with delay, 140
Odd functions, 92, 93
Odd number factorial, 198
One-dimensional heat equations, 210–211, 212
One-step transition probability, for Markov chains, 269
Open queues, 284
Operators, 190–191, 191–193. See also Difference operators
Laplace transforms as, 193–194
Order
of a difference equation, 221, 223
of a differential equation, 124, 203, 204
of a system, 78–79
Ordinary difference equations, 223–224
Ordinary differential equations (ODEs), 123, 124–127, 128, 143
applications of first-order, 159–164
applications of second-order, 199–203
Cauchy–Euler, 188–190
exact, 144–145
exactness test for, 147–148
Laplace transforms in solving, 193–194
miscellaneous methods for solving, 188–199
power series in solving, 195–199
second-order homogeneous, 164–175
second-order nonhomogeneous with constant coefficients, 175–188
separation of variables in solving, 158–159
solving by substitution method, 153–159
this book and, xi, xii
Ordinary points, analytic functions and, 195–196
Original function (original), of a Laplace transform, 94–95
Oscillation, in modeling population growth, 264
Oscillatory behavior, of delay logistic equation solutions, 139–140
Oscillatory systems, 199–203
Outcomes, 2
relative frequency of, 3
Outliers
skewness and, 59
in statistics, 57
Outputs, in queueing theory, 285
Overdamped spring-mass systems, 201, 202
Palm law, 284
Parabolic equations, 208, 209
Parallel computing systems, balking and reneging and, 308–309
Parallel queueing systems, 285, 312
Parallel queues, 257
queueing theory for multiserver, 311–325
Parallel queues with feedback, queueing theory for many-server, 326–328
Parallel stations, in queueing theory, 285
Parallel structure, 78–79
Parallel two-station single-server system, 329
Parameters
of binomial distribution function, 11
for gamma random variables, 29–30
in point estimation, 68
in regression analysis, 77
in statistics, 53
Parameter space, in stochastic processes, 267
Partial derivatives, 122, 123
exact differential equations and, 145–146, 147–148
in solving exact differential equations, 148–150, 152–153
notations for, 203
Partial difference equations, 244, 245
systems of, 249–251
Partial differences, 244
Partial differential equations (PDEs), 124
basic concepts of, 203–213
defined, 203
Fourier series and, 210–213
this book and, xi, xii
Partial differential equation systems, xii
Partial fraction(s) method, 193, 194, 250
Partially functioning state, 78
Particular solutions
of complementary equations, 176
of difference equations, 222, 232, 233, 234–235, 242
of nonhomogeneous differential equations, 177, 178–184, 184–188
Partitions, 2, 5
Pascal probability distribution function, 12–13
Percentiles, 58–59
with MINITAB software, 60–61
Percent relative frequency, in statistics, 54
Period, of a periodic Markov chain, 275
Periodic functions, Fourier series and, 90–91
Periodic Markov chains, 275
Poincaré, Henri, 218
Point estimates, 68
Point estimation, 68–72
interval estimation vs., 72
Point estimators, 68
Point functions, 20
Point processes, 267–268
Poisson distribution
for single processor with splitting and delayed feedback, 334–335, 336
with single-server Markovian queues, 286, 288
Poisson equation, 204
Poisson law, 284
Poisson parameter, estimating via MLE, 69–70
Poisson probability distribution, 27–28
probability generating function and, 113
table of values for, 358–360
Poisson probability distribution function, 15–16
in solving differential–difference equation systems, 256
Poisson process(es)
defined, 283–284
in finding stationary waiting time of a task, 324, 325
M/M/m queues with constant balking and exponential reneging and, 332
modeling differential–difference equation systems for, 254–259
in solving differential–difference equation systems, 257
Poisson random variables, 15–16
estimating parameter via MLE, 69–70
mean and variance of, 19–20
probability generating function and, 113
Poisson/Riemann transport equation, 204
Poisson single-processor model with splitter and feedback, 334
Poles of a system, 94, 107, 109, 110
Polya’s urn model, 273–275
Polynomial functions, for single processor with splitting and delayed feedback, 339
Population, variance of, 62–63. See also Populations
Population distributions, 58
in statistics, 58
Population growth, nonlinear difference equations in modeling, 260–264
Population growth/decay models, 129–130, 135–140, 218
Populations, 2–3
in statistics, 53, 58
Positive recurrent states, 275
Positive skew, 59
Power function, 76
Power of a test, 76
Power series, solving linear ODEs via, 195–199
Power series method, this book and, xii
Power sets, 20
Principle of superposition, 177, 181
Priority M/M/2 queues, with constant balking and exponential reneging, 328–332
Priority of receiving arrivals, 329
Priority service queue discipline, 285
Probabilistic arrivals, in queueing theory, 284
Probabilistic law, in queueing theory, 284
Probability, 1–89. See also n-steps
transition probability; Stationary transition probabilities; Statistics;
Transition probability
basic definitions and concepts of, 1–7
Bernoulli counting process and, 270–271
continuous probability distribution functions in, 26–41
continuous random variable moments in, 25–26
continuous random variables in, 20–25
continuous random vectors in, 48–49
Probability (cont’d)
convergence in, 50
difference–differential equations and, xi–xii
difference equations and, 219, 250–251
discrete random variable moments in, 16–20
discrete random variables in, 7–16
functions of a random variable in, 49–53
in gambler’s ruin problem, 228–229
generating functions method and, 245–246, 250–251
Kolmogorov axiomatization of, 21
Lebesgue measure and, 23
Polya’s urn model and, 273–275
probability distribution functions in, 7–16
random vectors in, 41–48
single-server Markovian queue with state-dependent balking and, 309–311
stationary queue length distribution and, 299
subsets of the real numbers and, 20
this book and, xi, 1
of a Type I error, 75–76
of a Type II error, 76
Probability density function(s) (pdf), 68
in Bernoulli trials, 11
continuous, 24
defined, 9–10
as Dirac delta function, 280–281
general logistic cumulative, 35–36, 37–38
of lognormal random variable, 35
probability distribution function and, 24–25
Probability distribution functions, 7–16, 68
continuous, 26–41
continuous random variables and, 24
geometric, 13
negative binomial (Pascal), 12–13
Poisson, 15–16
probability density function and, 24–25
Probability distributions. See also Busy period distribution
Cauchy, 26
discrete random variables and, 42
in finding stationary waiting time of a task, 323–325, 327–328
general logistic cumulative, 35–36, 37, 38
with $M/M/m$ queues with constant balking and exponential reneging, 333–33
of random variables, 9
standard logistic, 35, 36
tables of values for, 358–365
Weibull cumulative, 38–404
Probability generating functions (pgfs), 111–116
defined, 111–112
properties of, 112–116
with single-server Markovian queue with state-dependent balking, 310–311
in solving differential–difference equation systems, 256–259
Probability mass function (pmf), 68.
See also Joint probability mass functions (joint pmfs); Marginal probability mass functions (marginal pmfs);
Multinomial probability mass functions (multinomial pmfs)
in Bernoulli trials, 11, 12
defined, 9–10
geometric probability distribution function as, 13
moment generating functions and, 114–115
probability generating function and, 112
Probability measure, 23
Probability of an event, 3
axioms of, 3–4
Probability space, 3–4, 23
Probability theory, 1, 2
in statistics, 54, 55–58
Probability vectors, for Markov chains, 278
Pure birth processes, 254, 282–283
defined, 283–284
Pure death processes, 282–283

$p$-value, 76

Quadratic polynomials, in method of undetermined coefficients, 178–179

Quantification, 3

Quartiles, 58–59

Quasilinear partial differential equations (PDEs)
  first-order, 206–207
  second-order, 208–213

Queue disciplines, 285

Queueing models, this book and, xi

Queueing Models in Industry and Business (Haghighi & Mishev), 284

Queueing networks, 1

Queueing systems, 284–286
  as birth–death processes, 284–285
  parallel, 312
  single-server with splitting and feedback, 334–352

Queueing theory, 267–357
  basic concepts of, 267–268
  birth and death process in, 281–284
  for finite-buffer single-server Markovian queue, 303–307
  history and terminology of, 284–286
  for many-server parallel queues with feedback, 326–328
  for many-server queues with balking and reneging, 328–334
  Markov chains and processes in, 268–281
  for multiserver parallel queues, 311–325
  for single-server Markovian queue, 286–303
  for single-server Markovian queueing system with splitting and feedback, 334–352
  for single-server Markovian queue with feedback, 307–308
  for single-server Markovian queue with state-dependent balking, 308–311
  this book and, xi, xii

Queue length distribution
  for $M/M/1/N$ queueing system, 303–307
  for $M/M/m$ queues with constant balking and exponential reneging, 333–334

Queue lengths. See also Queue size
  algorithm for computing expected values of, 341–350, 351, 352
  with single-server Markovian queues, 287–291, 291–294, 297

Queues, 284–286

Queues, Inventories, and Maintenance (Morse), 284

Queue size, 284. See also Queue lengths
  with single-server Markovian queues, 287

Random counting processes, 268

Random delayed feedback, 350

Random error ($\varepsilon$), 77

Random experiments, 2

Randomness, probability and, 1–2

Random point processes, 268

Random samples/sampling, 3
  central limit theorem and, 64–67
  from a finite population, 58
  from an infinite population, 58
  likelihood function and, 68–69
  in statistics, 53, 57–58

Random service queue discipline, 285

Random time points, in stochastic processes, 268

Random variables
  Bernoulli, 19
  binomial, 11, 16
  central limit theorem and, 64–67
  Chebyshev’s inequality and, 51–52
  chi-square, 32
  coefficient of correlation of, 46–48
  continuous, 8, 20–25
  covariance of, 45–46
  defined, 8
  discrete, 7–16
  in finding stationary waiting time of a task, 323–325, 327–328
  first moment and, 17
  functions of, 49–53
  Galton, 35
  gamma, 29–30
  Gaussian, 32–33
Random variables (cont’d)
generating function of sums of
independent, 114
indicator, 9
lognormal, 35
Markov’s inequality and, 50–51
moment generating functions of
discrete, 114–115
moment of order \( r \) and, 17–18
moments of continuous, 25–26
moments of discrete, 16–20
mutually independent, 48–49
negative exponential, 28–29, 30
normal, 32–33
in point estimation, 68
Poisson, 15–16
probability distributions of, 9
properties of, 9
in random vectors, 41
standard normal, 33–34, 63
in statistics, 57–58
in stochastic processes, 267
uncorrelated, 46
variance and standard deviation of, 18
Weibull, 39
Random vectors, 41–48
continuous, 48–49
discrete, 41–42, 43
functions of random variables and, 49–50
in statistics, 57–58
Range, 3
of a function, 122
as measure of dispersion, 59–60
in statistics, 53
Rate of success, 271
Real-life mathematical models, 127–130
Real line (\( \mathbb{R} \)), 21–22
discrete random vectors and, 41
Lebesgue measure on, 23
Real numbers
measure and, 21
subsets of, 20, 21
Real roots, in solving \( n \)th-order linear
homogeneous difference equations, 225, 226, 230–231
Rectangular density functions, 27
Rectangular distribution functions, 27
Recurrent law, 284
Recurrent point processes, 268
Recurrent states, 275
Recursion, in solving nonlinear
difference equations, 259–260
Recursive formula for coefficients, 198
Recursive functions, 244–245
Recursive method
for multiserver parallel systems, 321
for single-server Markovian queue
with state-dependent balking, 310–311
in solving differential–difference
equations, 253, 255–256, 291
in solving differential–difference
equation systems, 255–256
in solving first-order linear
nonhomogeneous difference
equations, 231, 237–243
in solving \( n \)th-order linear
homogeneous difference equations, 225, 250
Reducible Markov chains, 275
Regeneration, in stochastic processes, 268
Region of convergence (\( ROC \)), of
\( Z \)-transforms, 105, 107, 108, 109, 110
Regression, 77
Regression analysis, 77
Regression coefficients, 77
Regular Markov chains, 275
Rejection region, 75
Relative frequency
of outcomes, 3
in statistics, 53–54, 55–58
Reliability, 78–81
Reliability function (\( R \)), 79–80, 81
Reliability of a system, 1, 2, 78–81
Remaining data set, 59
Reneging
many-server queues with, 328–334
\( M/M/m \) queues with exponential, 332, 333–334
priority \( M/M/2 \) queues with
exponential, 328–332
in queueing theory, 284, 308–309
Renewal counting processes, 268
Renewal periods, for stochastic
processes, 268
Renewal processes, 268
Representative samples, 3
Resistance–capacitor (RC) circuit, 121
Riemann–Hilbert problem, computing expected queue lengths and waiting times as, 342
Right skew, 59
Rouché theorem
M/M/1 transient queue length distribution and, 293
M/M/m transient queue length distribution and, 314
Rounding, 77–78
Rounding error, 77
Rounding interval, 77–78
rth central moment, 18
rth moment, 17–18
Sample mean, 58–59, 70
properties of, 61–62
Sample points, 3
Samples, 3. See also Random samples/sampling
Sample spaces, 2, 3–4, 5, 6–7
in Bernoulli trials, 10
continuous, 20
elements of, 7–8
equiprobable, 3
finite numbers of, 41
random variables and, 8
random vectors and, 41
real line as, 21–22
Sample standard deviation, 62
Sample statistics, properties of, 61–67
Sample variance, 60–61, 62–63
central limit theorem and, 64–67
Sampling, in statistics, 53. See also Random samples/sampling
Sampling distribution, 62
Scale parameter (μ), 31
of logistic distributions, 36–37
of Weibull cumulative probability distribution function, 38, 39, 40, 81
Scatter plots, in statistics, 57
Second moment, 18
Second-order difference equations, 124, 125, 128, 164–175, 175–188
applications of ordinary, 199–203
Cauchy–Euler, 188–190
initial value problems for, 167
linear nonhomogeneous with constant coefficients, 175–188
power series in solving, 195–199
solving linear homogeneous, 165–175
Second-order differential operators, 190–191
Second-order linear homogeneous difference equations, with constant coefficients, 225–226, 227–228
Second-order partial differential equations, 203–204, 208–213
Second quartile, 59
Second shifting, of Laplace transform, 100–103
Second shifting theorem, 101–102
Separable differential equations, 132–140, 160, 161
methods of solving, 133–140, 155
Separation of variables, 158–159
Sequences
defined, 220–221
nth-order linear homogeneous difference equations and, 224–225
in solving first-order linear nonhomogeneous difference equations, 240
stochastic processes and, 267–268
Z-transforms of, 106, 110–111
Series stations, in queueing theory, 285
Series structure, 78–79
Servers, in queueing theory, 285
Service completion
for multiserver parallel systems, 312, 313
with single-server Markovian queues, 288, 289–290
Service distribution, for single processor with splitting and delayed feedback, 335, 336
Service stations
in queueing theory, 285
for single processor with splitting and delayed feedback, 335, 336
Service times
for \( M/M/1/N \) queueing system, 303
with multiserver parallel systems, 312
in queueing theory, 285
with single-server Markovian queues, 287
Set functions, 20, 21
Sets, 2, 4. See also Borel sets
difference equations and, 221
functions and, 20
measurable, 21
nonmeasurable, 21
stochastic processes and, 267
Shape parameter, of Weibull cumulative
probability distribution function, 38, 39, 40
\( \sigma \)-algebra, 20, 21, 22
\( \sigma \)-field, 22
Signal processing, \( Z \)-transform in, 104
Signal processing theory, xii
Significance level of a test, 76
Simple events, 2
Simple exponential growth model, 129
Simple frequency distribution, in
statistics, 53–54, 55–58
Simulations, 81
Sine (\( \sin \)). See also Trigonometric
functions
Fourier series and, 91–94
Laplace transform of, 96–97
Single-parameter Weibull cumulative
probability distribution function, 38–39
Single processor with splitting and
delayed feedback, 334–352
Single-server Markovian (\( M/M/1 \)) queue
queueing theory for, 286–303
stationary queue length distribution
for, 294–299
transient queue length distribution for, 291–294
Single-server Markovian queue with
feedback, queueing theory for, 307–308
Single-server Markovian queue with
state-dependent balking, queueing
theory for, 308–311
Single servers, in queueing theory, 285
Singletons, 2
Singularities, analytic functions and, 195
Singular points, analytic functions and, 195–196, 197
Size of a class, in statistics, 57
Size of a population, in statistics, 58
Skewed data, 59
Skewness, in statistics, 57
Sojourn time
for many-server parallel queues with
feedback, 328
for \( M/M/1 \) queues, 300
Solution curves, in solving the logistic
differential equation, 137–140
Solutions
for difference equations, 222–223
for partial differential equations, 204–205
Special cases, in computing expected
queue lengths and waiting times, 349–350
Splitter, for single processor with
splitting and delayed feedback, 335
Splitting
in queueing theory, 285
single-server Markovian queueing
system with, 334–352
Spring–mass systems, 199–203
Springs, vibrating, 128, 199–203
s-shifting, of Laplace transform, 99
Stability, in computing expected queue
lengths and waiting times, 350–352
Stable cycles, in modeling population
growth, 260–261, 262
Stable equilibrium points, in modeling
population growth, 260–261, 262–263
Standard deviation, 18, 60–61, 62
central limit theorem and, 64–67
Standard error of the sample mean, 62
Standard logistic differential equation
with delay, 138–140
numerical solution for, 140
Standard logistic probability distribution, 35, 36
Standard normal confidence interval, 74
Standard normal density curve, 73
Standard normal (probability)
distribution, 73
central limit theorem and, 64–67
table of values for, 363–364
Standard normal random variables, 33–34, 63
Standard Weibull density, 38–39
Standard Weibull distribution, 38–39
State-dependent balking, $M/M/1$ queue with, 308–311
States
  of Markov chains, 275–276, 277
  of a queue, 284
State space ($S$)
  discrete-time stochastic processes with discrete, 269
  in stochastic processes, 267
State transition probabilities, for multisever parallel systems, 312–313
Stationary balance difference equations, for single processor with splitting and delayed feedback, 337–341
Stationary balance equations, for multisever parallel systems, 320–323
Stationary distribution of queue lengths for many-server parallel queues with feedback, 326–327
for $M/M/1/N$ queueing system, 303–307
Stationary distributions, of Markov chains, 277–278
Stationary initial probability distribution, of Markov processes, 280
Stationary Markov processes, 280
Stationary $M/M/1$ queue, 350
Stationary probability vectors, for Markov chains, 278
Stationary probability distribution, with $M/M/m$ queues with constant balking and exponential reneging, 333–334
Stationary processes, 278
Stationary queue length distribution for $M/M/1$ queues, 294–299
for multisever parallel systems, 320–323
Stationary transition probabilities, 257
Stationary waiting time, for $M/M/1$ queues, 300
Stationary waiting time of a task for many-server parallel queues with feedback, 327–328
for multisever parallel systems, 323–325
Stations, algorithm for computing expected queue lengths and waiting times at, 341–350, 351, 352
Statistical hypothesis, 75
Statistical inference, 67–68
Statistical populations, 2–3
Statistics, 1–89. See also Probability basic elements of, 53–67
descriptive, 1
difference–differential equations and, xi–xii
  hypothesis testing in, 67–68, 75–78
  inferential, 1, 67–75
  reliability in, 78–81
  this book and, xi, 1
Steady-state case, with single-server Markovian queues, 287
Steady-state distributions, of Markov chains, 277–278
Steady-state probability ($\Psi$) for single processor with splitting and delayed feedback, 336–341
in computing expected queue lengths and waiting times, 342–343
Steady-state probability vectors, for Markov chains, 278
Stem-an-leaves plots, in statistics, 57
Step functions, 100–101
Stieltjes, Thomas Joannes, 25
Stieltjes integral, 25
Stochastic analysis, xii
Stochastic convergence, 50
Stochastic matrices ($P$)
  for Markov chains, 269–270, 275–276, 277
  in Chapman–Kolmogorov equation, 272, 280
Stochastic processes, 49–50
defined/types of, 267
independent increments in, 278
for many-server parallel queues with feedback, 326
time of, 254
this book and, xii
Strength, 81
Stress, 81
Strings, vibrating, 209
Strong law of large numbers, 52–53
Structure function ($\Psi$), 78–79
Student's $t$-distribution, 74
  table of values for, 365
Student's $t$-distribution confidence interval, 74–75
Subsets, 2, 3
  in Borel fields, 22
  families of, 22
  measurable, 21
  of real numbers, 20, 21
Substitution, in solving nonlinear difference equations, 259–260
Substitution method, solving differential equations by, 153–159
Success, in Bernoulli trials, 10, 11, 12
Superposition principle, 177, 181
Survival, 80
Symbolism, in queueing theory, 285–286
Symmetry of data, in statistics, 57
Systems, reliability of, 1, 2, 78–81. See also Combined waiting and loss system; Critically damped spring–mass system; Differential–difference equation systems; Loss systems; Many-server queueing systems; Markovian queueing systems; $M/M/m$ entries; Multichannel queueing systems; Multiprocessor systems; Multiserver parallel systems; Oscillatory systems; Overdamped spring-mass systems; Parallel computing systems; Parallel queueing systems; Parallel two-station single-server system; Partial differential equation systems; Poles of a system; Queueing systems; Spring–mass systems; Tandem queueing systems; Waiting system; Zeros of a system

Takács’ waiting time distribution, 323
Tandem queue, for single processor with splitting and delayed feedback, 336
Tandem queueing systems, 285
Tasks
  for $M/M/m$ queues with constant balking and exponential reneging, 332
  with multiserver parallel systems, 312
in queues, 284
  waiting time of, 323–325, 327–328
Taylor series, analytic functions and, 195, 196
$t$-distribution, 74
Teaching, this book and, xi
Terminology, for this book, xii, xiii
Testing of a hypothesis, 75–78
Test statistic, 75
Third-order differential equations, 124
Third quartile, 58
Three-dimensional heat equation, 124
Threshold, for single processor with splitting and delayed feedback, 335
Time-dependent queue length distribution, for $M/M/1$ queues, 291–294
Time-homogeneous Markov processes, 280
Time-homogeneous transition probability density function (tpdf), 280–281
Time-independent queue length distribution, for $M/M/1$ queues, 294–299
Time points of renewal processes, 268
Total probability law, 5, 45
Traffic flow, for single-server Markovian queue with feedback, 308
Traffic intensity, for $M/M/1$ queues, 297, 299
Transforms, 90–120. See also Fourier transforms ($FT$); Laplace transforms this book and, xi
Transient case, with single-server Markovian queues, 287
Transient queue length distribution for $M/M/1$ queues, 291–294
  for multiserver parallel systems, 312–320
Transient states, 275
  busy period distribution and, 301
Transition probability, for Markov chains/processes, 269, 279
Transition probability density function (tpdf), of Markov processes, 280–281
Transition probability matrices \((P)\), 273, 275–276, 277, 280, 282
for busy periods, 301–302
for \(M/M/1/N\) queueing system, 303–304
with single-server Markovian queues, 289–290
Transport equations, 204, 209
Trapezoidal rule, 346–347, 347–348, 349
Traveling waves, 209
Trigonometric functions. See also Cosine (cos); Sine (sin)
Fourier series and, 90–91
\(M/M/1/N\) queueing system and, 305
Trigonometric series, 91
Trimmed mean, 59
Trimming of data, 59
Trivial Borel field, 22
t-shifting, of Laplace transform, 100–103
t-shifting theorem, 101–102
t-test, 76
Two-parameter Weibull cumulative probability distribution function, 38–40
Two-parameter Weibull distribution function, estimating via MLE, 70–72
Type 1 distribution, 40–41
Type I error, 75–76
Type II error, 75–76
Type 3 distribution, 40–41
Unbiased estimators, 64, 68
Uncorrelated random variables, 46
Underdamped spring–mass system, 201, 203
in solving Cauchy–Euler equations, 190
Uniform density function, 27
Uniform distribution, 26–28
Uniform distribution function, 26–28
Unilateral Z-transform, 105
probability generating function and, 112
Unions, 2, 4
of measurable sets, 21
Uniqueness of solutions of difference equations, 224, 238–239, 240
of differential equations, 130–132, 167, 175–176
Unit circle, 110, 258
Units, in queues, 284
Unit step function, 100–101
Unknown variance, 74–75
Urn model, 14, 15, 273–275. See also Ball and urn problems
Utility factor, for \(M/M/1\) queues, 297
Utilization factor, for \(M/M/1/N\) queueing system, 306
Variability, probability and, 2
Variables. See also Random variables; Separation of variables functions and differential equations and, 121, 122–130
notation for, xii
Variance \((\text{Var}, \sigma^2)\), 18, 61–62, 62–63. See also Analysis of variance (ANOVA);
Sample variance central limit theorem and, 64–67
generating functions method and, 248
of good estimators, 68
known, 72–74
of logistic distributions, 36
of a lognormal random variable, 35
moment generating functions and, 115–116
of normal random variables, 33
of a Poisson random variable, 19–20
probability generating function and, 113
of a random variable, 19
with single-server Markovian queue with state-dependent balking, 311
of a Weibull random variable, 39
unknown, 74–75
Variation of parameters method, 171
for nonhomogeneous differential equations, 184–188
Vector operators, 210
Vectors, notation for, xii
Verhulst, Pierre François, 129
Verhulst growth model, 129–130
Vibrating springs, 128, 199–203
Vibrating strings, 209
von Mises-Jenkinson-type distribution, 41
von Mises-type distribution, 41

Waiting buffer, for single processor with splitting and delayed feedback, 334–335
Waiting facility, in queueing theory, 285
Waiting line, with multiserver parallel systems, 312
Waiting rooms, in queueing theory, 285
Waiting system, in queueing theory, 285
Waiting time of a task for many-server parallel queues with feedback, 327–328
with multiserver parallel systems, 323–325
Waiting times, algorithm for computing expected values of, 341–350, 351, 352
Wave equations, 204, 208, 209
Wave phenomena, 209
partial differential equations for modeling, 203–204

Weak law of large numbers, 50, 52, 53
Weibull cumulative probability distribution function, 38–40
Weibull distribution, 81
Weibull parameter, estimating via MLE, 70–72
Weibull probability density, 39
Weibull random variables, 39
Weibull-type distribution, 40–41
Weight, 17
Weighted average, 17
Wronskian (W), 167, 169, 186

Yule law, 284

Zero balking rate, 309
Zeros of a system, 94, 107, 109
Z-transforms, 90, 104–111
defined, 105
in solving second-order linear nonhomogeneous difference equations with constant coefficients, 241–242
this book and, xii