Appendix B

Properties of Expectations
and Variances

Let \( Y \) denote a random variable that takes on values according to some probability density function if \( Y \) is continuous or some probability mass function if \( Y \) is discrete.

The expected value, or expectation, of \( Y \) is simply its mean or average value and is usually denoted by:

\[
E(Y) = \mu.
\]

It is often referred to as the first moment of \( Y \), since it describes the location of the center of the distribution. The precise definition of the expectation of \( Y \) is that it is a weighted average of all the possible values of \( Y \), with weights determined by the probabilities associated with each possible value.

The variance of \( Y \), often denoted by \( \sigma^2 \) or \( \text{Var}(Y) \), is a measure of the dispersion or variability around the mean or expected value of \( Y \). The variance is often referred to as the second central moment of \( Y \) and is defined as:

\[
\sigma^2 := \text{Var}(Y) := E[(Y - E(Y))^2].
\]

The variance is a weighted average of the squared deviations of \( Y \) around its mean. Because the variance is expressed in squared units of \( Y \), a measure of variability in
the original units of \( Y \) is given by the standard deviation,
\[
\sigma := \sqrt{\text{Var}(Y)}.
\]

Finally, the covariance between two random variables, \( X \) and \( Y \), is defined as
\[
\text{Cov}(X, Y) := \mathbb{E} \left[ [X - \mathbb{E}(X)][Y - \mathbb{E}(Y)] \right],
\]
and is a measure of the linear dependence between \( X \) and \( Y \). If \( X \) and \( Y \) are independent, then \( \text{Cov}(X, Y) = 0 \). Note that the covariance of a variable with itself is the variance, \( \text{Cov}(Y, Y) = \text{Var}(Y) \).

Properties of Expectations and Variances

Next we consider some properties of expectations and variances. Let \( X \) and \( Y \) be two (possibly dependent) random variables and let \( a \) and \( b \) denote non-random constants.

Then the expectation operator, \( \mathbb{E}(\cdot) \), has the following five important properties:

1. \( \mathbb{E}(a) = a \)
2. \( \mathbb{E}(bX) = b \mathbb{E}(X) \)
3. \( \mathbb{E}(a + bX) = a + b \mathbb{E}(X) \)
4. \( \mathbb{E}(aX + bY) = a \mathbb{E}(X) + b \mathbb{E}(Y) \)
5. \( \mathbb{E}(XY) = \mathbb{E}(X) \mathbb{E}(Y) \) (unless \( X \) and \( Y \) are independent)

Thus expectation is a linear operator in the sense that it respects or preserves the arithmetic operations of addition and multiplication by a constant. As a result the expected value of a linear function of \( Y \) (e.g., \( a + bY \)) is simply the same linear function of the expected value of \( Y \) (e.g., \( a + b \mathbb{E}(Y) \)).

The variance operator, \( \text{Var}(\cdot) \), has the following five important properties:

1. \( \text{Var}(a) = 0 \)
2. \( \text{Var}(bX) = b^2 \text{Var}(X) \)
3. \( \text{Var}(a + bX) = b^2 \text{Var}(X) \)
4. \( \text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab \text{Cov}(X, Y) \)
5. \( \text{Var}(aX - bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) - 2ab \text{Cov}(X, Y) \)

In particular, if \( X \) and \( Y \) are dependent, then
\[
\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2 \text{Cov}(X, Y)
\]
and
\[
\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) - 2 \text{Cov}(X, Y).
\]
Finally, we note that the expectation and variance operators can also be applied to vectors of random variables. For example, let \( \mathbf{Y} \) be a \( n \times 1 \) (column) response vector (e.g., repeated measurements at \( n \) different occasions),

\[
\mathbf{Y} = \begin{pmatrix}
Y_1 \\
Y_2 \\
\vdots \\
Y_n
\end{pmatrix},
\]

then:

\[
\mathbf{E}(\mathbf{Y}) = \begin{pmatrix}
\mathbf{E}(Y_1) \\
\mathbf{E}(Y_2) \\
\vdots \\
\mathbf{E}(Y_n)
\end{pmatrix},
\]

and:

\[
\text{Cov}(\mathbf{Y}) = 
\begin{pmatrix}
\text{Var}(Y_1) & \text{Cov}(Y_1, Y_2) & \ldots & \text{Cov}(Y_1, Y_n) \\
\text{Cov}(Y_2, Y_1) & \text{Var}(Y_2) & \ldots & \text{Cov}(Y_2, Y_n) \\
\vdots & \vdots & \ddots & \vdots \\
\text{Cov}(Y_n, Y_1) & \text{Cov}(Y_n, Y_2) & \ldots & \text{Var}(Y_n)
\end{pmatrix}.
\]