Appendix C

Shock Wave Propagation in a Two-Dimensional Flow Field

The basic characteristics of a one-dimensional shock wave are described in Chapter 1 of this text. However, the shock waves in supersonic flow propagate not only one-dimensionally but also two- or three-dimensionally in space. For example, the shock waves formed at the air-intake of a ducted rocket are two- or three-dimensional in shape. Expansion waves are also formed in supersonic flow. The pressure downstream of an expansion wave is reduced and the flow velocity is increased. With reference to Chapter 1, brief descriptions of the characteristics of a two-dimensional shock wave and of an expansion wave are given here.[1−5]

C.1 Oblique Shock Wave

The formation of a shock wave is dependent on the objects that affect the flow field. The conservation of mass, momentum, and energy must be satisfied at any location. This is manifested in the formation of a shock wave at a certain location in the flow field to meet the conservation equations. In the case of a blunt body in a supersonic flow, the pressure increases in front of the body. The increased pressure generates a detached shock wave to satisfy the conservation equations in the flow field to match the conserved properties between the inflow and outflow in front of the body. The velocity then becomes a subsonic flow behind the detached shock wave. However, the shock wave distant from the blunt body is less affected and the detached shock wave becomes an oblique shock wave. Thus, the shock wave appears to be curved in shape, and is termed a bow shock wave, as illustrated in Fig. C-1.

Fig. C-2(a) shows an attached shock wave on the tip of a wedge. This is a weak shock wave formed when the associated pressure difference is small. On the other hand, as shown in Fig. C-2(b), a detached shock wave is formed when the pressure difference becomes large. An attached shock wave becomes a detached shock wave when the wedge angle becomes large.

When a two-dimensional wedge is placed in a supersonic flow, a shock wave that
propagates from the tip of the wedge is formed. Unlike a normal shock wave, the streamline is not perpendicular to the shock wave, and this is termed an oblique shock wave. As shown in Fig. C-3, the shock wave is deflected by an angle, $\beta$, and the streamline is also inclined at an angle, $\theta$. The velocity along the streamline changes from $w_1$ to $w_2$ through the oblique shock wave. The velocity component perpendicular to the shock wave changes from $u_1$ to $u_2$ and the velocity component parallel to the shock wave changes from $v_1$ to $v_2$. The velocity triangle shown in Fig. C-3 is expressed by

$$ w_2^2 = u_2^2 + v_2^2 \tag{C.1} $$

Though the velocity component parallel to the shock wave remains unchanged, $v_1 = v_2$, the velocity component normal to the shock wave, $u_1 \to u_2$, changes through the shock wave. The change in the normal velocity component through the oblique
shock wave is equivalent to the velocity change through the normal shock wave. Then, $u_1$ shown in Fig. C-3 is equivalent to $u_1$ shown in Fig. 3.1 and the Rankine–Hugoniot relationship between pressure and density for an oblique shock wave becomes equivalent to that for a normal shock wave given by Eqs. (C.2) and (C.3) as follows:

$$p_2/p_1 = \frac{(\rho_2/\rho_1)\zeta - 1}{(\zeta - \rho_2/\rho_1)} \quad \text{(C.2)}$$

$$\rho_2/\rho_1 = \frac{(\rho_2/\rho_1)\zeta + 1}{(\rho_2/\rho_1 + \zeta)} \quad \text{(C.3)}$$

where $\zeta$ is given by $\zeta = (\gamma + 1)/(\gamma - 1)$.

The angle between the inflow streamline and the oblique shock wave, $\beta$, is expressed by

$$\beta = \tan^{-1}(u_1/v_1) \quad \text{(C.4)}$$

Since the Mach number of the inflow to the shock wave is given by $M_1 = w_1/a_1$ and that of the outflow from the shock wave is given by $M_2 = w_2/a_2$, the Mach number of the normal velocity perpendicular to the oblique shock wave, $M_1^\perp$, is represented by

$$u_1/a_1 = M_1 \sin \beta = M_1^\perp \quad \text{(C.5)}$$

Thus, the oblique shock-wave equations are obtained by replacing $M_1$ with $M_1^\perp$ in the normal shock-wave equations, Eqs. (3.19)–(3.23), as follows:

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma + 1} M_1^{\perp 2} \frac{1}{\zeta} \quad \text{(C.6)}$$

$$\frac{p_{02}}{p_{01}} = \left[ \frac{(\gamma + 1)M_1^{\perp 2}}{(\gamma - 1)M_1^{\perp 2} + 2} \right]^{\frac{\gamma - 1}{\gamma - 1} - \frac{2\gamma}{\gamma + 1}} \left( \frac{2\gamma}{2\gamma + 1} (M_1^{\perp 2} - 1) + 1 \right)^{\frac{1}{\gamma - 1}} \quad \text{(C.7)}$$

$$M_1^\perp = \left[ \frac{\gamma + 1}{2\gamma} \frac{p_2 - p_1}{p_1} + 1 \right]^{\frac{1}{2}} \quad \text{(C.8)}$$
The entropy change through the oblique shock wave is given by

\[ s_2 - s_1 = c_p \ln \left( \frac{T_2}{T_1} \right) - R_g \ln \left( \frac{p_2}{p_1} \right) \]  

(1.45)

Since \( p_2/p_1 \geq 1 \) in Eq. (C.8), the Mach number normal to the oblique shock wave is

\[ M_1^{*2} \geq 1 \]  

(C.11)

and then one obtains the relationship

\[ \sin^{-1}(1/M_1) \leq \beta \leq \pi/2 \]  

(C.12)

The incline angle along the streamline of the upstream defined by

\[ \alpha = \sin^{-1}(1/M_1) \]  

(C.13)

is termed the Mach angle, as defined in Chapter 2. It is shown the angle of the oblique shock wave, \( \beta \), is larger than the Mach angle, \( \alpha \).

The angle \( \eta \), between the shock-wave angle, \( \beta \), and the angle of the streamline behind the shock wave, \( \theta \), given by

\[ \eta = \beta - \theta \]  

(C.14)

is represented by

\[ \tan \eta = u_2/v_2 \]  

(C.15)

Combining Eqs. (C.1), (C.2), and (C.8) with the relationships \( v_1 = v_2 \) and \( u_2/u_1 = \rho_1/\rho_2 \), one obtains

\[ \tan \eta/\tan \beta = 2/\{ (\gamma + 1) M_1^{*2} \} + 1/\zeta \]  

(C.16)

It is shown that two \( \beta \) and two \( \theta \) correspond to one \( M_1 \). When \( \beta \) is small, the static pressure ratio \( p_2/p_1 \) is small, and the shock wave is weak. On the other hand, when \( \beta \) is large, a strong shock wave is formed, for which \( p_2/p_1 \) is large. The Mach number behind the oblique shock wave becomes supersonic for weak shock waves and subsonic for strong shock waves.

Based on Eq. (C.15), the Mach number \( M_1^{*} \) is obtained as

\[ M_1^{*} = \{ (\gamma + 1)(M_1^{*2}/2) \sin \beta \sin \theta / \cos \zeta + 1 \}^{1/2} \]  

(C.17)
When \( \theta \) is small, Eq. (C.17) becomes
\[
M_1^* = \left\{ (\gamma + 1)(M_1^2/2) \theta \tan \theta + 1 \right\}^{1/2}
\]
(C.18)

C.2 Expansion Wave

Let us consider a supersonic flow along a wall surface with a corner of negative angle \((-\theta)\). The flow is governed by the same conservation equations as for an oblique shock wave formed along a wall surface with a corner of positive angle \((+\theta)\). The key difference is that an oblique shock wave is formed when the corner has a positive angle whereas an expansion wave is formed when the corner has a negative angle.

The expansion wave is formed in a fan-shape with the corner at its center, as shown in Fig. C-4. The expansion wave consists of a multitude of Mach waves. The first Mach wave, with an angle of \(\alpha_1\), is formed at the front-end of the expansion wave, and the last Mach wave, with an angle of \(\alpha_2\), is formed at the rear-end of the expansion wave, these being represented by \(\alpha_1 = \sin^{-1}(1/M_1)\) and \(\alpha_2 = \sin^{-1}(1/M_2)\), respectively.

In the expansion wave, the flow velocity is increased and the pressure, density, and temperature are decreased along the stream line through the expansion fan. Since \(\alpha_1 > \alpha_2\), it follows that \(M_1 < M_2\). The flow through an expansion wave is continuous and is accompanied by an isentropic change known as a Prandtl–Meyer expansion wave. The relationship between the deflection angle and the Mach number is represented by the Prandtl–Meyer expansion equation.[1–5]

C.3 Diamond Shock Wave

When a supersonic flow emerges from a rocket nozzle, several oblique shock waves and expansion waves are formed along the nozzle flow. These waves are formed repeatedly and form a brilliant diamond-like array, as shown in Fig. C-5. When an under-expanded flow, i.e., having pressure \(p_e\) higher than the ambient pressure \(p_a\), is formed at the nozzle exit, an expansion wave is formed to decrease the pressure. This expansion wave is reflected at the interface between the flow stream and the ambient air and a shock wave is formed. This process is repeated several times to form a diamond array, as shown in Fig. C-6 (a).

![Figure C-4. Expansion wave formed in supersonic flow along a wall surface with a corner of negative angle.](image)
a) Under expansion

On the other hand, an over-expanded flow is formed at a nozzle exit when the pressure \( p_s \) is lower than that of the ambient atmosphere \( p_a \), and a shock wave is formed to increase the pressure. This shock wave is reflected at the interface between the flow stream and the ambient air and an expansion wave is formed. As in the case of the under-expanded flow, this process is repeated several times to form a diamond array, as shown in Fig. C-6 (b).

References