Simplified lattice Boltzmann method for non-Newtonian power-law fluid flows

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Summary
In this paper, we present a simplified lattice Boltzmann method for non-Newtonian power-law fluid flows. The new method adopts the predictor-corrector scheme and reconstructs solutions to the macroscopic equations recovered from the lattice Boltzmann equation through Chapman-Enskog expansion analysis. The truncated power-law model is incorporated into this method to locally adjust the physical viscosity and the associated relaxation parameter, which recovers the non-Newtonian behaviors. Compared with existing non-Newtonian lattice Boltzmann models, the proposed method directly evolves the macroscopic variables instead of the distribution functions, which eliminates the intrinsic drawbacks like high cost in virtual memory and inconvenient implementation of physical boundary conditions. The validity of the method is demonstrated by benchmark tests and comparisons with analytical solution or numerical results in the literature. Benchmark solutions to the three-dimensional lid-driven cavity flow of non-Newtonian power-law fluid are also provided for future reference.

KEYWORDS
lattice Boltzmann method, non-Newtonian fluid, power-law model, viscosity

1 | INTRODUCTION

Non-Newtonian fluid refers to a specific category of fluid that exhibits variable viscosity under the action of force. It certainly differs from the Newtonian fluid that follows Newton's law of viscosity and bears constant viscosity under stress. The physical viscosity in non-Newtonian fluids could be dependent on the magnitude of the shear stress (e.g., shear-thickening/dilatant fluids, shear-thinning/pseudoplastic fluids, Bingham plastics, etc) or even time-dependent (e.g., thixotropic liquids, rheopectic liquids, etc). In fact, many practical fluids may present these non-Newtonian behaviors. Typical examples include blood, silicone oils, printer ink, polymers, etc.

Numerical interpretation of the non-Newtonian behaviors thus becomes an important topic from both the academic and the application points of view. Explorations in this topic have been continuing for over a century; and many mathematical models have been constructed, which include the generalized Newtonian models, the linear or elementary nonlinear viscoelastic models, the models with memory-integral expansion, etc. Specifically, the generalized Newtonian fluid models maintain the general form of governing equations for Newtonian fluid while making the viscosity dependent on the shear rate.1 The linear viscoelastic models, which were initially proposed by Maxwell,2 essentially establish a linear differential relationship between the stress tensor and the shear rate. Later, the elementary nonlinear viscoelastic model tracks the evolution of a particular fluid particle and thus consolidates the nonlinear convection effect into the constitutive relation, which makes the model more “objective.”3,4 More recently, some models were proposed by the memory-integral expansion strategy,5,6 which assumes that the stress in a fluid element is only dependent on its own kinetic history and
can thus be achieved by integrations over time. Among these approaches of constructing non-Newtonian models, the
generalized Newtonian fluid models seem to be the simplest and the most straightforward way due to the similarity to the
Newtonian expressions. Moreover, the most commonly used one in this category might be the power-law model. In this
model, the local viscosity is assumed to follow an exponential relationship with the shear rate. By adjusting the power
index, typical non-Newtonian behaviors like the shear-thinning or the shear-thickening can be interpreted. In this work,
the power-law model will be adopted to study non-Newtonian fluid flow.

The non-Newtonian models are essential constitutive relations and should be consolidated into specific equations that
govern the fluid system. One popular mathematical interpretation is the lattice Boltzmann equation (LBE) or the lattice
Boltzmann method (LBM), which is a mesoscopic method developed on the kinetic theory. In LBM, the distribution
function is the primitive variable that is directly evolved, and the conservative variables and fluxes at the macroscopic
scale can be obtained by integrating moments of the distribution functions in the velocity space. The LBM seems to
be more attractive than conventional Navier-Stokes (N-S) solvers in some aspects such as the kinetic nature, simplicity,
and explicitness. Being constructed at the mesoscopic scale, LBM is a more general model than the N-S equations in
describing fluid behaviors. Its evolution of distribution functions is essentially accomplished in the velocity space rather
than in the physical space, and the explicitness of LBM formulations facilitates its implementation, especially in parallel
computations.

Efforts have been made in incorporating the non-Newtonian models into the LBM to recover and investigate this distinct
physical phenomenon. The earliest attempt might be carried out by Aharonov and Rothman who used the conventional
single-relaxation-time (SRT) lattice Boltzmann model as the flow solver. Gabbanelli et al recognized that the SRT model
might be unstable at extreme values of the relaxation parameter and thus implemented a truncated power-law model
that sets upper and lower limits to the fluid viscosity. Giraud et al. and Sullivan et al. both explored the simulations
in three-dimensional scenarios. Yoshino et al. proposed an alternative model that decouples the relaxation parameter
from the viscosity by consolidating the shear stress into the equilibrium distribution functions. Also, this model was
then modified by Wang et al. to ensure mass conservation. Apart from the SRT model, a few lattice Boltzmann models
with multiple relaxation time (MRT) were also proposed for simulations of non-Newtonian fluid flows. Other
than developing non-Newtonian lattice Boltzmann schemes, abundant examples of applying lattice Boltzmann models
in investigations of non-Newtonian phenomena can also be found in literature.

Although being an important solution candidate for non-Newtonian problems, the lattice Boltzmann models may not
be the mainstream solution to this category of problems. Limitations of LBM, eg, high cost in virtual memory, incon-
vienent implementation of physical boundary conditions, and poor numerical stability in small viscosity scenario, are
somehow intrinsic and hard to be removed. The first two limitations both originate from the cornerstone of LBM,
which is the evolution of distribution functions. The number of distribution functions at each mesh point is much larger
than the number of macroscopic variables. Therefore, more virtual memory would be consumed. In the meantime, the
physical boundary conditions that are related to the macroscopic variables should be converted to boundary conditions
for the distribution functions as to be implemented in the conventional lattice Boltzmann framework. Such transforma-
tion is tedious: specific strategies should be tailored for various physical boundary conditions, and the implementa-
tions on complex geometries are sometimes challenging. In addition, the stability analysis suggested that the LBM with a single
relaxation parameter would become unstable when the relaxation parameter approaches 0.5, which is equivalent
to the condition of small physical viscosity. Some amending strategies have been proposed to alleviate the stability issue,
which include the MRT model, the entropic lattice Boltzmann schemes, the regularized LBE, etc. However, all these attempts are carried out within the conventional framework and thus cannot remove the first two
limitations of LBM.

The simplified LBM (SLBM) is a recent exploration towards eliminating or alleviating limitations of the conven-
tional LBM. Instead of directly evolving the distribution functions, SLBM reconstructs solutions to the macroscopic
equations recovered from LBE through the Chapman-Enskog (C-E) expansion analysis. The second-order accuracy is
preserved in the reconstruction process, and the final formulations reflect direct evolution of the macroscopic variables.
Consequently, the dependent variables of distribution functions in the conventional LBM are replaced by the macro-
sopic flow variables in the SLBM, which reduces the number of variables at each mesh point (see Table 1); the physical
boundary conditions can be implemented straightforwardly, and the flexibility of the method on nonuniform or bodyfit-
ted meshes is enriched. Moreover, the von Neumann stability analysis indicates that SLBM is stable at all wavenumbers
in high Reynolds number scenario, which makes it a competitive candidate for real engineering problems.

The idea of SLBM has been applied in various fluid problems such as the thermal flows, fluid-structure interaction,
axisymmetric flows, multiphase flows, etc. Yet its extension to the non-Newtonian simulations has not been reported.
Although the idea is similar to its Newtonian counterpart, detailed derivations and the final formulations of the method for non-Newtonian flows would differ since the physical viscosity would be coupled with the flow field. In this paper, we will present the SLBM for non-Newtonian power-law fluids and give comprehensive evaluation of its performance in simulations.

The remaining parts of this paper are organized as follows. Section 2 briefs the LBM and the power-law model for non-Newtonian fluids. Further explanation on the truncated power-law model and detailed derivation of its association with the SLBM are presented in Section 3. Numerical validations of the proposed method are carried out in Section 4, based on which benchmark solutions to some problems are given. Finally, Section 5 remarks the conclusions of this paper.

## 2 | LATTICE BOLTZMANN METHOD AND THE POWER-LAW MODEL

### 2.1 | Lattice Boltzmann method

The LBM resolves the flow field by updating the distribution function \( f_\alpha \), which follows the LBE as shown in the following:

\[
f_\alpha (r + e_\alpha \delta_t, t + \delta_t) = f_\alpha (r, t) + \frac{f^\text{eq}_\alpha (r, t) - f_\alpha (r, t)}{\tau},
\]

where \( r \) and \( t \) are the location vector and the time level, respectively; \( \tau \) is the single relaxation parameter in the BGK collision model; \( e_\alpha \) is the lattice velocity vector along the \( \alpha \)-th direction; \( \delta_t \) represents the time interval; and the superscript \( \text{eq} \) denotes the equilibrium state of the distribution function

\[
f^\text{eq}_\alpha = \rho \omega_\alpha \left[ 1 + \frac{e_\alpha \cdot u}{c_s^2} + \frac{(e_\alpha \cdot u)^2 - (c_s |u|)^2}{2 c_s^4} \right].
\]

Here, \( \rho \) and \( u \), respectively, denote the density and the velocity vector, and \( \omega_\alpha \) and \( c_s \) are the weighting coefficient and the sound speed, respectively. For a typical D2Q9 model, we have

\[
e_\alpha = \begin{cases} 0 & \alpha = 0, \\ (\pm 1, 0) & \alpha = 1, 2, 3, 4, \\ (\pm 1, \pm 1) & \alpha = 5, 6, 7, 8, \end{cases}
\]

\[
\omega_0 = \frac{4}{9}, \quad \omega_{1\sim4} = \frac{1}{9}, \quad \omega_{5\sim8} = \frac{1}{36}, \quad c_s = \frac{c}{\sqrt{3}}.
\]

Moreover, the D3Q15 model gives

\[
e_\alpha = \begin{cases} 0 & \alpha = 0, \\ (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1) & \alpha = 1 \sim 6, \\ (\pm 1, \pm 1, \pm 1) & \alpha = 7 \sim 14, \end{cases}
\]

\[
\omega_0 = \frac{2}{9}, \quad \omega_{1\sim6} = \frac{1}{9}, \quad \omega_{7\sim14} = \frac{1}{72}, \quad c_s = \frac{c}{\sqrt{3}},
\]

where \( c = \delta_x / \delta_t \) is the lattice velocity, and \( \delta_x \) is the lattice spacing.
2.2 | Chapman-Enskog expansion analysis

The LBE is believed to be an interpretation of more general fluid behaviors spanning from the continuum regime to the slip flow regime. Specifically, in the continuum flow regime, LBE is consistent to the macroscopic Navier–Stokes equations with desired order of accuracy. Such consistency can be constructed through the C-E expansion analysis.

The C-E analysis of the LBE starts from the multiscale expansions of the distribution function, temporal, and spatial derivatives in terms of a parameter $\varepsilon$ that is proportional to the Knudsen number, which gives

$$f_a = f_a^{(0)} + \varepsilon f_a^{(1)} + \varepsilon^2 f_a^{(2)},$$

$$\frac{\partial f_a}{\partial t} = \varepsilon \frac{\partial f_a^{(0)}}{\partial t_0} + \varepsilon^2 \frac{\partial f_a^{(1)}}{\partial t_1},$$

$$\nabla_r = \varepsilon \nabla_r^{(1)}.$$  

Performing Taylor-series expansion on Equation (1) yields

$$\left( \frac{\partial}{\partial t} + e_a \cdot \nabla \right) f_a + \frac{\delta t}{2} \left( \frac{\partial}{\partial t} + e_a \cdot \nabla \right)^2 f_a + \frac{1}{\tau_0} \left( f_a - f_a^{eq} \right) + O \left( \delta t^2 \right) = 0.$$  

Substituting Equations (7) to (9) into the above equation and consolidating obtained formulations at different scales, one can derive the following macroscopic equations:

$$\rho \frac{\partial f_a}{\partial t} + \nabla \cdot \left( \sum_a e_a f_a^{eq} \right) = 0,$$

$$\rho \frac{\partial f_a^{eq}}{\partial t} + \nabla \cdot \sum_a \left( e_a \rho (e_a) \right) \left[ f_a^{eq} + \left( 1 - \frac{1}{2\varepsilon} \right) f_a^{neq} \right] = 0.$$  

where

$$f_a^{neq} = -\tau \delta t D f_a^{eq}.$$  

Here, $f_a^{eq} = f_a^{(0)}$, $f_a^{neq} = \varepsilon f_a^{(1)}$, and $D = \left( \frac{\partial}{\partial t} + e_a \cdot \nabla \right)$.

2.3 | The power-law model for non-Newtonian fluids

For non-Newtonian fluids, the physical viscosity is dependent on the local shear rate through the constitutive equation. A simple and widely used model for non-Newtonian fluid is the power-law model, which evaluates the local viscosity through

$$\mu = \mu_0 |\dot{\gamma}|^{n-1},$$

where $\mu = \rho v$ is the dynamic viscosity, $\mu_0$ is the flow consistency coefficient, and $n > 0$ denotes the power law index. Specifically, cases with $n < 1$ and $n > 1$, respectively, represent the shear-thinning (pseudoplastic) and the shear-thickening (dilatant) fluids, while the setting of $n = 1$ recovers regular Newtonian fluid behaviors. The shear rate $\dot{\gamma}$ is defined as

$$\dot{\gamma} = \nabla u + (\nabla u)^T,$$

while its magnitude is calculated by

$$|\dot{\gamma}| = \sqrt{(\dot{\gamma} : \dot{\gamma})/2}.$$  

Correspondingly, the relaxation parameter should be locally adjusted when simulating non-Newtonian fluid flows. In addition, the local value of the relaxation parameter is obtained by substituting the local viscosity into Equation (14).
3 | SIMPLIFIED LATTICE BOLTZMANN METHOD FOR NON-NEWTONIAN POWER-LAW FLUIDS

3.1 | Formulations

Following the idea of SLBM for incompressible Newtonian fluid flows, the derivation of the present model for non-Newtonian fluid flows starts from the macroscopic governing Equations (11) and (12) recovered from LBE. Implementing the fractional step technique, the recovered equations can be resolved in the following predictor-corrector strategy:

Predictor step:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \left( \sum_a \rho \mathbf{e}_a f^eq \right) \mathbf{f} = 0, \]  
\[ \frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \left( \sum_a \rho \mathbf{e}_a \gamma \left[ f^eq + \frac{1}{2} f^neq \right] \right) \mathbf{f} = 0. \]  

Corrector step:

\[ \frac{\partial \rho}{\partial t} = 0, \]  
\[ \frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \left( \sum_a \rho \mathbf{e}_a \gamma \left[ 1 - \frac{1}{\tau} \right] f^neq \right) \mathbf{f} = 0. \]

The essential part of the derivation is to reconstruct solutions to these equations within the lattice Boltzmann framework. Specifically, in the predictor step, the intermediate flow variables are proposed to be

\[ \rho^* = \sum_a f^eq \left( \mathbf{r} - \mathbf{e}_a \delta_t, t - \delta_t \right), \]  
\[ \rho^* \mathbf{u}^* = \sum_a \mathbf{e}_a f^eq \left( \mathbf{r} - \mathbf{e}_a \delta_t, t - \delta_t \right). \]

Performing the Taylor-series expansion analysis on the above formulations and applying the relationships given by C-E analysis (Equation (13)), it can be proven that, to the second-order accuracy, Equations (22) and (23) recover solutions to the macroscopic Equations (18) and (19) in the predictor step, respectively. Detailed proofs for this step can be found in the Appendix.

In the corrector step, as shown in Equation (20), no correction would be made on the intermediate density. To facilitate the derivation of the corrected velocity, Equation (21) is rewritten into the following form with the help of Equation (13):

\[ \frac{\partial \rho \mathbf{u}}{\partial t} - \nabla \cdot \left( \sum_a \rho \mathbf{e}_a \gamma f^eq \right) \mathbf{f} = 0, \]  
where

\[ f^eq = (\tau - 1) \delta_t D f^eq. \]

Moreover, its solution can be reconstructed by

\[ \rho^{n+1} \mathbf{u}^{n+1} = \rho^* \mathbf{u}^* + \sum_a \mathbf{e}_a \left[ f^- (\mathbf{r} + 0.5 \mathbf{e}_a \delta_t, t - 0.5 \delta_t) - f^- (\mathbf{r} - 0.5 \mathbf{e}_a \delta_t, t - 0.5 \delta_t) \right]. \]

The consistency between Equations (26) and (24) can be established through Taylor series expansion. It can be demonstrated that, to the second-order accuracy in space and time, Equation (26) is equivalent to Equation (24). For reference, detailed derivations for this step are attached in the Appendix.

The remaining issue here is the approximation of the newly defined quantity \( f^- \) at specific space and time levels. Since the recovery of Equation (24) from Equation (26) involves the space derivative of the second moment of \( f^- \), the approximated \( f^- \) would be divided by the mesh spacing \( \delta_x \), which is equal to \( \delta_t \). Thus, the approximation of \( f^- \) should fulfill the third-order accuracy of \( O(\delta_t^3) \) in order not to deteriorate the overall second-order accuracy. To fulfill that requirement, the approximated \( f^- \) locates at the space/time level of \( \left( \mathbf{r} \pm 0.5 \mathbf{e}_a \delta_t, t - 0.5 \delta_t \right) \). In the meantime, given the fact that the marching in time levels is from \( t - \delta_t \) to \( t \), the shifting of half time step of \( f^- \) in Equation (26) facilitates establishing a
central difference approximation, which ensures the second order of temporal accuracy, and the following formulations can be derived:

\[
f_a^c (r \pm 0.5e_a \delta_t, t - 0.5\delta_t) = \left[ \tau (r \pm 0.5e_a \delta_t, t - 0.5\delta_t) - 1 \right] \delta_t D_{a}^{eq} f_a^c (r \pm 0.5e_a \delta_t, t - 0.5\delta_t),
\]

\[
\tau (r \pm 0.5e_a \delta_t, t - 0.5\delta_t) = \frac{\tau (r, t) + \tau (r \pm e_a \delta_t, t) + \tau (r, t - \delta_t) + \tau (r \pm e_a \delta_t, t - \delta_t)}{4},
\]

\[
\delta_t D_{a}^{eq} f_a^c (r \pm 0.5e_a \delta_t, t - 0.5\delta_t) = f_a^{eq} (r \pm e_a \delta_t, t) - f_a^{eq} (r, t - \delta_t) + O (\delta_t^3),
\]

\[
\delta_t D_{a}^{eq} f_a^{eq} (r \pm 0.5e_a \delta_t, t - 0.5\delta_t) = f_a^{eq} (r, t) - f_a^{eq} (r - e_a \delta_t, t - \delta_t) + O (\delta_t^3).
\]

Note that the intermediate properties obtained in the predictor step are utilized here as the physical properties at the time level of \( t \).

Compared with conventional lattice Boltzmann models, the derived SLBM can expect some intriguing merits while maintaining advantages of the conventional LBM such as the kinetic nature, simplicity, and explicitness. The final formulations of SLBM reflect evolution of macroscopic variables instead of the distribution functions. The equilibrium distribution function can be evaluated by the macroscopic properties, while the nonequilibrium distribution function is approximated by the difference of two equilibrium distribution functions that are also associated with the macroscopic variables. Thus, the minimum number of variables at each point for SLBM is the number of macroscopic flow properties (3 in two-dimensional cases). Even if storing the intermediate flow properties, the total number of variables at each mesh point is 6. In contrast, the dependent variables in the conventional LBM are the distribution functions. Therefore, the minimum number of variables at each mesh point is the number of lattice velocity directions (9 variables for the D2Q9 model). Therefore, compared with the conventional LBM, 33.3% of virtual memory can be reduced by using the present SLBM with the D2Q9 model. In Table 1, the theoretical reduction rates of virtual memory for different lattice velocity models are provided. It should be noted that the expected reduction rates provided here are only for reference. The practical memory cost would be dependent on the programming skills and the compromise with computational efficiency.

In the meantime, by using the macroscopic flow variables as the dependent variables, the physical boundary conditions that are directly associated with the macroscopic flow variables can be implemented in a more straightforward way in the present SLBM. The tedious transformation into conditions for the distribution functions in the conventional LBM is thus removed. Moreover, the stability analysis of SLBM for Newtonian fluids reveals that it is more stable than conventional lattice Boltzmann models, which is a favored characteristic for non-Newtonian simulations in which fluid viscosity exhibits large variations.

### 3.2 Truncated power-law model

Direct implementation of the power-law model could lead to serious issues due to its unphysical limits at zero shear rates. Referring to Equations (15) to (17), at zero shear rates, the power-law model would give infinite viscosity for shear-thinning fluids while yielding zero viscosity for shear-thickening fluids. In the meantime, these two unphysical limits also pose numerical challenges for LBM. Within the lattice Boltzmann framework, fluid viscosity is reflected by the relaxation parameter \( \tau \). The large viscosity and the inviscid condition correspond to a large value of \( \tau \) and \( \tau = 0.5 \), respectively. Previous studies indicated that the conventional LBM would become highly unstable when \( \tau \) approaching 0.5, and show deteriorated numerical accuracy for \( \tau \geq 1 \). To alleviate this issue, Gabbanelli et al. proposed the truncated power-law model. By setting upper and lower limits for the fluid viscosity (\( \nu_{max} = 0.1 \) and \( \nu_{min} = 0.001 \) in their paper), the corresponding relaxation parameters vary within desired ranges (\( \tau_{max} = 0.8 \) and \( \tau_{min} = 0.503 \)).

In the present study, we follow the similar way of adopting truncated power-law model, but directly implement limits for the relaxation parameter. As shown in the previous stability analysis for Newtonian fluid, SLBM bears very good numerical stability in the extreme case of \( \tau = 0.5 \). Therefore, the lower limit is set as \( \tau_{min} = 0.5 \), and the upper limit of the relaxation parameter is \( \tau_{max} = 1.2 \) in all numerical examples, if not particularly specified. The larger adjustable range of the relaxation parameter could endow more flexibility when recovering non-Newtonian behaviors. Note that the local shear rate in Equation (16) is calculated by the finite difference method.
3.3 | Implementation of boundary conditions

As mentioned before, the evolution of macroscopic variables in SLBM allows direct implementation of physical boundary conditions without tedious transformation into boundary conditions for the distribution functions. For Dirichlet boundary conditions, boundary values of the physical properties can be directly assigned on the boundary mesh points. Moreover, normally, information on two inner layers of mesh points is utilized to evaluate boundary values when implementing Neumann boundary conditions, which ensures the second-order accuracy in space.

It is noteworthy that the corrector step of SLBM requires the boundary values of the newly defined quantity $f\sim\alpha$. Recognizing that $f\sim\alpha$ is continuous over the physical space, the linear extrapolation scheme is adopted here to determine its boundary value. This strategy has been adopted in the simulation of Newtonian fluid flows.\(^{30}\) Given the fact that this quantity is a minor term in the order of $O(\delta_t)$, the linear extrapolation scheme would not deteriorate the global second-order accuracy.

3.4 | Computational sequence

For illustration purposes, one possible procedure of implementing the proposed SLBM for non-Newtonian power law fluid is presented as follows:

1. Mesh initialization: Specify the streaming distance $\delta_x (\delta_x = \delta_t)$.
2. Predictor step: Use Equations (22) and (23) to evaluate intermediate flow properties $\rho^*$ and $u^*$.
3. Calculate the nonequilibrium term $f^\sim\alpha$ using Equations (27) to (30).
4. Implement linear extrapolation scheme to determine boundary values of $f^\sim\alpha$.
6. Implement appropriate physical boundary conditions for the macroscopic properties.
7. Repeat steps (2) to (6) until the computation is converged or the prescribed maximum iteration step is reached.

4 | NUMERICAL EXAMPLES

In this part, numerical tests on the plain Poiseuille flow, the two-dimensional lid-driven cavity flow, and the three-dimensional lid-driven cavity flow are carried out. The accuracy, convergence, and robustness of the proposed method are demonstrated by comparing the computational results with analytical solution or reference data in literature. Moreover, some benchmark solutions for the three-dimensional lid-driven cavity flow of power-law fluids are provided by the validated method.
FIGURE 2  Velocity profiles along centerlines in two-dimensional lid-driven cavity flow for non-Newtonian power-law fluid at Re = 100 and 500. Left: X-Y curve; Right: U-Y curve

4.1 | Plain Poiseuille flow

The proposed method is first tested on the plain Poiseuille flow of non-Newtonian power-law fluids. The physical configuration of this example is the flow between two parallel plates respectively placed at \( y = \pm H/2 \). The flow is driven by a constant pressure gradient \( \partial p/\partial x \) along the streamwise direction and would exhibit a unidirectional velocity profile along the transverse direction at the steady state, which gives

\[
    u(y) = \frac{n}{n+1} \left( \frac{1}{\mu_0} \frac{\partial p}{\partial x} \right)^{1/n} \left[ \left( \frac{H}{2} \right)^{1+1/n} - |y|^{1+1/n} \right].
\]  

(31)

Moreover, the maximum velocity is

\[
    u_{\text{max}} = \frac{n}{n+1} \left( \frac{1}{\mu_0} \frac{\partial p}{\partial x} \right)^{1/n} \left( \frac{H}{2} \right)^{1+1/n}.
\]  

(32)
Numerical simulations are carried out on the uniform mesh size of $41 \times 41$, and various power law indexes of $n = 0.25$, 0.5, 1.0, 1.5, and 2.0 are considered. Specifically, no-slip boundary condition is implemented on the upper and lower walls, while periodic boundary condition is adopted on the inlet and outlet boundaries. Computational results are presented in Figure 1 and compared with the analytical solutions. The good agreements between the analytical and the numerical solutions at different power-law indexes provide initial validation of the accuracy of the proposed SLBM for non-Newtonian power-law fluids. However, this example essentially reflects a one-dimensional fluid behavior; validations on more complex problems are necessary.

### 4.2 Two-dimensional lid-driven cavity flow

The two-dimensional lid-driven cavity flow has been a widely used benchmark test for numerical schemes. Here, we use this problem to further validate the accuracy and robustness of SLBM. The lid-driven cavity flow refers to the fluid system in a square cavity with its top lid moving at a constant velocity $U_0$. As a typical internal viscous flow, solutions to this problem are characterized by the Reynolds number. Specifically, for non-Newtonian power-law fluid, the Reynolds number is defined as

$$
Re = \frac{\rho U_0^{2-n} L^n}{\mu_0}.
$$

(33)
FIGURE 4  Streamlines of the two-dimensional lid-driven cavity flow for non-Newtonian power-law fluid

TABLE 2  Comparisons of the location of the primary vortex in lid-driven cavity flow of non-Newtonian power-law fluids with $n = 0.5$

<table>
<thead>
<tr>
<th></th>
<th>$Re = 100$</th>
<th>$Re = 500$</th>
<th>$Re = 5000$</th>
<th>$Re = 7500$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_c$</td>
<td>Present</td>
<td>0.7118</td>
<td>0.5697</td>
<td>0.5282</td>
</tr>
<tr>
<td></td>
<td>Li et al$^{15}$</td>
<td>0.7168</td>
<td>0.5793</td>
<td>0.5255</td>
</tr>
<tr>
<td>$y_c$</td>
<td>Present</td>
<td>0.7815</td>
<td>0.5501</td>
<td>0.5131</td>
</tr>
<tr>
<td></td>
<td>Li et al$^{15}$</td>
<td>0.7826</td>
<td>0.5497</td>
<td>0.5190</td>
</tr>
</tbody>
</table>

where $L$ is the length of the square cavity.

To reveal the non-Newtonian behaviors, we consider two different power indexes of 0.5 and 1.5, which respectively correspond to the shear-thinning and shear-thickening fluids. For this example, the effect of the moving lid is transmitted through the shear forces within the boundary layers. The shear-thinning fluid would further reduce the physical viscosity within the boundary layer, which yields thinner boundary layer and higher velocity gradient near the top lid. Such flow characteristic would pose extra difficulty to numerical simulations. Therefore, the mesh resolutions adopted in our simulations aim to capture the sharp velocity gradient changes in the shear-thinning cases.

Simulations are first conducted on uniform mesh sizes of $201 \times 201$ for cases at low Reynolds numbers of 100 and 500. The velocity profiles along the centerlines are plotted in Figure 2 and compared with reference values in literature.$^{14,36}$ Note that the velocity and the physical location are normalized by the $U_0$ and $L$, respectively. The computational results obtained by SLBM agree well with the reference data, which validates its accuracy for this example. Specifically, as shown in the U-Y curves, the velocity gradient near the top lid in the case of $n = 0.5$ is significantly larger than the gradient in the case of $n = 1.5$. This can be explained by the shear-thinning behavior mentioned above.

At higher Reynolds numbers, velocity gradients near the boundaries would be further enlarged. Therefore, the nonuniform meshes are preferred in such circumstances. The implementation of SLBM on nonuniform meshes relies on appropriate interpolation algorithm to determine the physical properties at streaming nodes. In the present study, we adopt the three-point Lagrange interpolation algorithm whose effectiveness has been validated in previous tests of
Newtonian flows. Validation is performed in the case of $\text{Re} = 5000$ on nonuniform mesh size of $201 \times 201$, and the velocity profiles along centerlines are presented in Figure 3. The good agreement between our numerical results and the reference data\textsuperscript{15,38} demonstrates the correctness of our implementation on nonuniform meshes and the validity of SLBM at higher Reynolds numbers. Also enclosed in Figure 3 are the convergence test and benchmark solution at $\text{Re} = 7500$. Here, we adopted four sets of mesh sizes, i.e., $41 \times 41$, $201 \times 201$, $301 \times 301$, and $401 \times 401$, in numerical simulations. The computational results in the last three sets of meshes are quite close, which proves the convergence of SLBM at this Reynolds number. Given its validated accuracy and robustness in previous cases, numerical results in the finest mesh size can be referred to as the benchmark solution to the problem at $\text{Re} = 7500$. Another interesting point is that smooth velocity profiles are obtained on the coarse mesh size of $41 \times 41$. For conventional LBM, it is quite challenging to give stable solution on such mesh size at high Reynolds numbers. Also, the smallest relaxation parameter in this particular case reads $\tau_{\text{min}} = 0.500001$, which is very much likely to induce numerical instability for conventional LBM.\textsuperscript{23} Such performance suggests that the present SLBM bears better numerical stability than conventional lattice Boltzmann models.

Streamlines at different Reynolds numbers and power law indexes are shown in Figure 4. The smoothness of the streamlines indicates the numerical stability of SLBM in simulating non-Newtonian power-law fluid flows. It is also noted that the shear-thickening fluid ($n = 1.5$) is more likely to trigger secondary minor vortices than the shear-thinning fluid ($n = 0.5$) at the right-bottom corner. This is caused by the stagnation point at the right-bottom corner which induces small shear rates nearby. For shear-thickening fluid, the small shear rates would reduce the physical viscosity and thus stimulates the generation of minor vortices. The shear-thinning fluid, on the contrary, exhibits large physical viscosity at small shear rates, which hinders the emerging of minor vortices. Quantitative comparisons of the location of the primary vortex for shear-thinning cases are carried out in Table 2. Good agreement between the present results and the reference data\textsuperscript{15} further validates the accuracy of the proposed method in this test example.

### 4.3 Three-dimensional lid-driven cavity flow

Although numerical examples of two-dimensional non-Newtonian power-law fluids are abundant in literature, relative studies in three-dimensional scenario remain scarce. Here, we present the computational results of SLBM for three-dimensional lid-driven cavity flow. Similar to the two-dimensional lid-driven cavity flow, solutions to this three-dimensional problem are also determined by the Reynolds number defined in Equation (33). Since the accuracy and robustness of SLBM have been validated in previous examples, here we aim to provide converged solutions at various Reynolds numbers. These converged solutions could be utilized as benchmark results in future studies.

Three different Reynolds numbers of 400, 1000, and 1500 are considered here. Two power-law indexes of $n = 0.5$ and 1.5, which respectively correspond to the shear-thinning and shear-thickening flows, are studied. The computational domain is illustrated in Figure 5, and convergence tests are carried out in a series of nonuniform mesh sizes of $61^3$, $81^3$, $101^3$, and $121^3$. Figures 6 and 7 present velocity profiles along the centerlines in cases with $n = 0.5$ and 1.5, respectively. For cases of $\text{Re} = 400$ and 1000, little variation of the velocity profile is observed when the mesh size is finer than $81^3$. For the
FIGURE 6  Velocity distributions along the centerlines in three-dimensional lid-driven cavity flow for non-Newtonian power-law fluid with $n = 0.5$. Left: $X-U_z$ curve; Right: $U_x-Z$ curve
FIGURE 7  Velocity distributions along the centerlines in three-dimensional lid-driven cavity flow for non-Newtonian power-law fluid with $n = 1.5$. Left: $X-U_z$ curve; Right: $U_x-Z$ curve
case of Re = 1500, convergence is achieved at the finer mesh size of 101³. Specifically, the converged results reveal larger velocity gradients near the top lid in shear-thinning scenario, which is consistent to the flow characteristics reflected in the two-dimensional simulations as shown in Section 4.2.

To further illustrate the physical flow pattern, streamlines on the mid-plane of y = 0.5 are shown in Figure 8. It is noted that the streamlines are more squeezed to the top lid in shear-thinning fluid (n = 0.5) due to the high velocity gradient, which is consistent to the velocity profile presented in Figure 6. The emerging of minor vortices at the bottom corners is more stimulated in the shear-thickening scenario (n = 1.5), which can be explained by reduced physical viscosity at lower
shear rate in such condition. In general, the numerical results fulfill the expectation and can be well explained from the physical perspective.

5 | CONCLUSIONS

An SLBM for non-Newtonian power-law fluid flows is proposed in this paper. This method extracts the essential idea of its Newtonian counterpart and reconstructs solutions to the macroscopic equations recovered from the LBE through the C-E expansion analysis. The reconstruction process follows a predictor-corrector manner and utilizes isotropic properties and relationships given in the C-E expansion analysis. To model the non-Newtonian behaviors in which the physical viscosity varies in the space, a truncated power-law model is incorporated to locally adjust the relaxation parameter in the method. Compared with other non-Newtonian lattice Boltzmann models, the present one tracks the evolution of macroscopic variables instead of distribution functions, which reduces the cost of virtual memory and facilitates implementation of physical boundary conditions.

The accuracy and robustness of the proposed method is validated by two examples of the plain Poiseuille flow and two-dimensional lid-driven cavity flow with power-law fluids. Further tests on three-dimensional lid-driven cavity flow of shear-thinning and shear-thickening fluids are then carried out, which validates the convergence of the method and provides benchmark solutions for future reference.

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**APPENDIX**

**DETAILED DERIVATION OF THE FORMULATIONS OF SLBM**

The formulations of SLBM in the predictor step are Equations (22) and (23). Performing Taylor series expansion on the equilibrium distribution functions with respect to the space/time level of \((\mathbf{r}, t)\) gives

\[
 f_a^{eq}(\mathbf{r} - \mathbf{e}_a \delta_t, t - \delta_t) = f_a^{eq}(\mathbf{r}, t) - \delta_t D f_a^{eq}(\mathbf{r}, t) + \frac{\delta_t^2}{2} D^2 f_a^{eq}(\mathbf{r}, t) + O(\delta_t^3),
\]

where \(D = \left( \frac{\partial}{\partial t} + \mathbf{e}_a \cdot \nabla \right)\). Substituting the above equation into Equations (22) and (23), and utilizing the relationship in Equation (13) yield

\[
 \rho^* = \sum_a f_a^{eq}(\mathbf{r}, t) - \delta_t \sum_a D f_a^{eq}(\mathbf{r}, t) + \frac{\delta_t^2}{2} \sum_a D^2 f_a^{eq}(\mathbf{r}, t) + O(\delta_t^3)
\]

\[
 = \rho(\mathbf{r}, t) - \delta_t \left[ \frac{\partial}{\partial t} \sum_a f_a^{eq}(\mathbf{r}, t) + \nabla \cdot \sum_a \mathbf{e}_a f_a^{eq}(\mathbf{r}, t) \right] - \delta_t \sum_a D \frac{1}{2\tau} f_a^{neq}(\mathbf{r}, t) + O(\delta_t^3),
\]

\(A2\)
\[
\rho^* \mathbf{u}^* = \sum_a \mathbf{e}_a f_a^\text{eq} (\mathbf{r} - \mathbf{e}_a \delta_t, t - \delta_t) \\
= \sum_a \mathbf{e}_a f_a^\text{eq} (\mathbf{r}, t) - \delta_t \sum_a \mathbf{e}_a D f_a^\text{eq} (\mathbf{r}, t) + \frac{\delta_t^2}{2} \sum_a \mathbf{e}_a D^2 f_a^\text{eq} (\mathbf{r}, t) + O (\delta_t^3) \\
= \rho (\mathbf{r}, t) \mathbf{u} (\mathbf{r}, t) - \delta_t \left[ \frac{\partial}{\partial t} \sum_a \mathbf{e}_a f_a^\text{eq} (\mathbf{r}, t) + \nabla \cdot \sum_a (\mathbf{e}_a)_p (\mathbf{e}_a)_r f_a^\text{eq} (\mathbf{r}, t) \right] \\
- \delta_t \sum_a \mathbf{e}_a D \frac{1}{2\tau} f_a^{\text{neq}} (\mathbf{r}, t) + O (\delta_t^3). \\
\]

(A3)

Note that the compatibility condition gives

\[
\sum_a f_a^{\text{neq}} = \sum_a \mathbf{e}_a f_a^{\text{neq}} = 0. \\
\]

(A4)

We can thus derive

\[
\sum_a D \frac{1}{2\tau} f_a^{\text{neq}} = \sum_a \frac{\partial}{\partial t} \left( \frac{1}{2\tau} f_a^{\text{neq}} \right) + \sum_a \mathbf{e}_a \cdot \nabla \left( \frac{1}{2\tau} f_a^{\text{neq}} \right) \\
= \frac{\partial}{\partial t} \left( \frac{1}{2\tau} \sum_a f_a^{\text{neq}} \right) + \nabla \cdot \left( \frac{1}{2\tau} \sum_a \mathbf{e}_a f_a^{\text{neq}} \right) \\
= 0, \\
\]

\[
\sum_a \mathbf{e}_a D \frac{1}{2\tau} f_a^{\text{neq}} (\mathbf{r}, t) = \sum_a \frac{\partial}{\partial t} \left( \frac{1}{2\tau} \mathbf{e}_a f_a^{\text{neq}} (\mathbf{r}, t) \right) + \sum_a \mathbf{e}_a \cdot \nabla \left( \frac{1}{2\tau} f_a^{\text{neq}} (\mathbf{r}, t) \right) \\
= \frac{\partial}{\partial t} \left( \frac{1}{2\tau} \sum_a \mathbf{e}_a f_a^{\text{neq}} \right) + \nabla \cdot \sum_a (\mathbf{e}_a)_p (\mathbf{e}_a)_r \left( \frac{1}{2\tau} f_a^{\text{neq}} \right) \\
= \nabla \cdot \sum_a (\mathbf{e}_a)_p (\mathbf{e}_a)_r \left( \frac{1}{2\tau} f_a^{\text{neq}} \right). \\
\]

(A6)

Therefore, Equations (A2) and (A3) can be reduced to

\[
\rho^* = \rho (\mathbf{r}, t) - \delta_t \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot \sum_a \mathbf{e}_a f_a^{\text{eq}} (\mathbf{r}, t) + O (\delta_t^2) \right], \\
\]

(A7)

\[
\rho^* \mathbf{u}^* = \rho (\mathbf{r}, t) \mathbf{u} (\mathbf{r}, t) - \delta_t \left[ \frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \sum_a (\mathbf{e}_a)_p (\mathbf{e}_a)_r \left( f_a^{\text{eq}} (\mathbf{r}, t) + \frac{1}{2\tau} f_a^{\text{neq}} (\mathbf{r}, t) \right) + O (\delta_t^2) \right], \\
\]

(A8)

which means that Equations (22) and (23) recover Equations (18) and (19) with second-order accuracy.

Similarly, the evolution equation of SLBM in the corrector step (ie, Equation (26)) can be expanded as

\[
\frac{\rho^{n+1} \mathbf{u}^{n+1} - \rho^* \mathbf{u}^*}{\delta_t} = \frac{1}{\delta_t} \sum_a \mathbf{e}_a \left[ f_a^+ (\mathbf{r} + 0.5 \mathbf{e}_a \delta_t, t - 0.5 \delta_t) - f_a^- (\mathbf{r} - 0.5 \mathbf{e}_a \delta_t, t - 0.5 \delta_t) \right] \\
= \frac{1}{\delta_t} \sum_a \mathbf{e}_a \left[ \nabla \cdot \mathbf{e}_a \delta_t f_a^+ (\mathbf{r}, t - 0.5 \delta_t) + O (\delta_t^2) \right] \\
= \nabla \cdot \sum_a (\mathbf{e}_a)_p (\mathbf{e}_a)_r f_a^+ (\mathbf{r}, t - 0.5 \delta_t) + O (\delta_t^2). \\
\]

(A9)

It is demonstrated that the Equation (26) recovers the space derivative term in the macroscopic Equation (24) with second-order accuracy. Moreover, the approximated term is at the time level of \((t-0.5\delta_t)\), which reflects the central difference scheme in time marching and thus ensures second-order accuracy both in time and in space.