Bohr radius for locally univalent harmonic mappings

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Abstract
We consider the class of all sense-preserving harmonic mappings $f = h + \overline{g}$ of the unit disk $\mathbb{D}$, where $h$ and $g$ are analytic with $g(0) = 0$, and determine the Bohr radius if any one of the following conditions holds:

1. $h$ is bounded in $\mathbb{D}$.
2. $h$ satisfies the condition $\text{Re } h(z) \leq 1$ in $\mathbb{D}$ with $h(0) > 0$.
3. both $h$ and $g$ are bounded in $\mathbb{D}$.
4. $h$ is bounded and $g'(0) = 0$.

We also consider the problem of determining the Bohr radius when the supremum of the modulus of the dilatation of $f$ in $\mathbb{D}$ is strictly less than 1. In addition, we determine the Bohr radius for the space $B$ of analytic Bloch functions and the space $B_H$ of harmonic Bloch functions. The paper concludes with two conjectures.

KEYWORDS
Bloch space, Bohr radius, harmonic, locally univalent, and analytic functions, $K$-quasiconformal mappings, Schwarz lemma

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1 | INTRODUCTION AND PRELIMINARIES

We shall investigate Bohr’s radius for complex-valued harmonic mappings and locally univalent harmonic mappings defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The Bohr theorem about power series (after subsequent improvements due to M. Riesz, I. Schur and F. Wiener) states that if $f$ is a bounded analytic function on $\mathbb{D}$, with the Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\sum_{n=0}^{\infty} |a_n| r^n \leq \|f\|_{\infty}$$

for $0 < r \leq 1/3$ and the constant $1/3$ is sharp. The best constant $r$ in the above inequality, which is $1/3$, is called the Bohr radius for the class of all analytic self-maps of the unit disk $\mathbb{D}$. The original problem goes back to 1914’s. Many mathematicians have contributed toward the understanding of this problem in several settings. We refer to the recent survey on this topic by Abu-Muhanna et al. [1] for the importance, background, and several other recent results and extensions. For certain recent results, see [2,11,12].

A harmonic mapping in $\mathbb{D}$ is a complex-valued function $f = u + iv$ of $z = x + iy$ in $\mathbb{D}$, which satisfies the Laplace equation $\Delta f = 4f_{zz} = 0$, where $f_z = (1/2)(f_x - if_y)$ and $f_{\overline{z}} = (1/2)(f_x + if_y)$ and where $u$ and $v$ are real-valued harmonic functions on $\mathbb{D}$. It follows that $f$ admits the canonical representation $f = h + \overline{g}$, where $h$ and $g$ are analytic in $\mathbb{D}$ with $f(0) = h(0)$.
The Jacobian $J_f$ of $f$ is given by $J_f = |h'|^2 - |g'|^2$. We say that $f$ is sense-preserving in $\mathbb{D}$ if $J_f(z) > 0$ in $\mathbb{D}$. Consequently, $f$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $J_f(z) > 0$ in $\mathbb{D}$; or equivalently if $h' \neq 0$ in $\mathbb{D}$ and the dilatation $\omega_f = \frac{g'}{h'}$ has the property that $|\omega(z)| < 1$ in $\mathbb{D}$ (see [14]).

In order to state the first result about Bohr radius for quasiconformal harmonic mappings, we need to introduce some notation. For harmonic mappings $f$ in $\mathbb{D}$, we use the following standard notations:

$$\Lambda_f(z) = \max_{0 \leq \theta \leq 2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = |f_z(z)| + |f_{\bar{z}}(z)|$$

and

$$\lambda_f(z) = \min_{0 \leq \theta \leq 2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = ||f_z(z)| - |f_{\bar{z}}(z)||$$

so that if $f$ is locally univalent and sense-preserving, then

$$J_f = \lambda_f \Lambda_f = |f_z|^2 - |f_{\bar{z}}|^2 > 0.$$

A sense-preserving homeomorphism $f$ from the unit disk $\mathbb{D}$ onto $\Omega'$, contained in the Sobolev class $W^{1,2}_{loc}(\mathbb{D})$, is said to be a $K$-quasiconformal mapping if, for $z \in \mathbb{D}$,

$$\frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} = \frac{1 + |\omega_f(z)|}{1 - |\omega_f(z)|} \leq K, \quad \text{i.e.,} \quad |\omega_f(z)| \leq k = \frac{K - 1}{K + 1},$$

where $K \geq 1$ so that $k \in [0, 1)$ (cf. [13, 16]).

**Theorem 1.1.** Suppose that $f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$ is a sense-preserving $K$-quasiconformal harmonic mapping of the disk $\mathbb{D}$, where $h$ is a bounded function in $\mathbb{D}$. Then

$$\sum_{n=0}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \leq ||h||_\infty \quad \text{for} \quad r \leq \frac{K + 1}{5K + 1}.$$  

The constant $(K + 1)/(5K + 1)$ is sharp.

**Theorem 1.2.** Assume the hypothesis of Theorem 1.1. Then

$$|a_0|^2 + \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \leq ||h||_\infty \quad \text{for} \quad r \leq \frac{K + 1}{3K + 1}.$$  

The constant $(K + 1)/(3K + 1)$ is sharp.

We would like to remark that the boundedness condition on $h$ in Theorem 1.1 can be replaced by half-plane condition. However, the Bohr radius remains the same in this case too.

**Theorem 1.3.** Suppose that $f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$ is a sense-preserving $K$-quasiconformal harmonic mapping of the disk $\mathbb{D}$, where $h$ satisfies the condition $\text{Re} h(z) \leq 1$ in $\mathbb{D}$ and $h(0) = a_0$ is positive. Then

$$a_0 + \sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \leq 1 \quad \text{for} \quad r \leq \frac{K + 1}{5K + 1}.$$  

The constant $(K + 1)/(5K + 1)$ is sharp.

The following corollaries are regarded as harmonic analogs of the classical Bohr inequality and will be of independent interest. These results are obtained by allowing $K \to \infty$ in the above results.

**Corollary 1.4.** Suppose that $f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$ is a sense-preserving harmonic mapping of the disk $\mathbb{D}$, where $h$ is a bounded function in $\mathbb{D}$. Then

$$|a_0| + \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \leq ||h||_\infty \quad \text{for} \quad r \leq \frac{1}{5}, \quad (1.1)$$
and the number $1/5$ is sharp. Moreover, either $a_0 = 0$ or $|a_0|$ in (1.1) is replaced by $|a_0|^{1/2}$, then the constant $1/5$ could be replaced by $1/3$ which is also sharp.

**Corollary 1.5.** Suppose that $f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$ is a sense-preserving harmonic mapping of the disk $\mathbb{D}$, where $h$ satisfies the condition $\text{Re} \, h(z) \leq 1$ in $\mathbb{D}$ and $h(0) = a_0$ is positive. Then

$$
a_0 + \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \leq 1 \quad \text{for} \quad r \leq \frac{1}{5},$$

and the number $1/5$ is sharp.

**Theorem 1.6.** Suppose that either $f = h + g$ or $f = h + \overline{g}$, where $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are bounded analytic functions in $\mathbb{D}$. Then

$$
\sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \leq \max \{ ||h||_{\infty}, ||g||_{\infty} \} \quad \text{for} \quad r \leq \sqrt{\frac{7}{32}},
$$

This number $\sqrt{7/32}$ is sharp.

As in the symmetric case of analytic functions (see [2,11,12]), we have the following analog result for harmonic functions.

**Theorem 1.7.** Let $p \geq 2$. Suppose that $f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^{pn+1} + \sum_{n=0}^{\infty} b_n z^{pn+1}$ is a harmonic $p$-symmetric function in $\mathbb{D}$, where $h$ and $g$ are bounded functions in $\mathbb{D}$. Then

$$
\sum_{n=1}^{\infty} (|a_n| + |b_n|) r^{pn+1} \leq \max \{ ||h||_{\infty}, ||g||_{\infty} \} \quad \text{for} \quad r \leq \frac{1}{2},
$$

The number $1/2$ is sharp.

The proofs of these results will be given in Section 2. In Section 3, we extend further results for sense-preserving $K$-quasiconformal harmonic mappings of the disk $\mathbb{D}$. In Section 4, we consider the problem of finding the Bohr radius for the space of bounded harmonic Bloch functions. The paper concludes with a couple of conjectures.

## 2 \ THE PROOFS OF THEOREMS 1.1, 1.2, 1.3, 1.6 AND 1.7

The following lemma is needed for the proof our first theorem.

**Lemma 2.1.** Suppose that $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are two analytic functions in the unit disk $\mathbb{D}$ such that $|g'(z)| \leq k|h'(z)|$ in $\mathbb{D}$ and for some $k \in [0, 1]$. Then

$$
\sum_{n=1}^{\infty} |b_n|^2 r^n \leq k^2 \sum_{n=1}^{\infty} |a_n|^2 r^n \quad \text{for} \quad |z| = r < 1.
$$

**Proof.** We integrate inequality $|g'(z)|^2 \leq k^2 |h'(z)|^2$ over the circle $|z| = r$ and get

$$
\sum_{n=1}^{\infty} n^2 |b_n|^2 r^{2(n-1)} \leq k^2 \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2(n-1)}.
$$

We integrate the last inequality with respect to $r^2$ and obtain

$$
\sum_{n=1}^{\infty} n |b_n|^2 r^{2n} \leq k^2 \sum_{n=1}^{\infty} n |a_n|^2 r^{2n}.
$$

One more integration (after dividing by $r^2$) gives desired inequality. \qed
Proof of Theorem 1.1. For simplicity, we suppose that $|h|_{\infty} = 1$. Then $|a_n| \leq 1 - |a_0|^2$ for $n \geq 1$. Let $\omega$ denote the dilatation of $f = h + g$ so that $|g'(z)| \leq k|h'(z)|$ in $\mathbb{D}$, where $k \in [0, 1)$ and so, by Lemma 2.1, it follows that
\[
\sum_{n=1}^{\infty} |b_n|^2 r^n \leq k^2 \sum_{n=1}^{\infty} |a_n|^2 r^n \leq k^2 (1 - |a_0|^2)^2 \frac{r}{1-r}.
\]
Consequently,
\[
\sum_{n=1}^{\infty} |b_n| r^n \leq \sqrt{\sum_{n=1}^{\infty} |b_n|^2 r^n} \sqrt{\sum_{n=1}^{\infty} r^n} \leq k (1 - |a_0|^2) \frac{r}{1-r}
\]
so that
\[
\sum_{n=0}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \leq |a_0| + (1 - |a_0|^2)(1 + k) \frac{r}{1-|a|r}
\]
which is clearly less than or equal to 1 for $r \leq 1/(3 + 2k)$. Substituting $k = (K - 1)/(K + 1)$ gives the desired result.

To prove the sharpness, consider
\[
h(z) = \frac{a - z}{1 - \overline{a}z} = a + \sum_{n=1}^{\infty} a_n z^n, \quad a_n = -(1 - |a|^2)(\overline{a})^{n-1} \text{ for } n \geq 1,
\]
and $g(z) = \lambda k h(z)$, where $|\lambda| = 1, a \in \mathbb{D}$ and $k = (K - 1)/(K + 1)$. Then it is a simple exercise to see that
\[
\sum_{n=0}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n = |a| + (1 - |a|^2)(1 + k) \sum_{n=1}^{\infty} |a|^{n-1} r^n = |a| + (1 - |a|^2)(1 + k) \frac{r}{1-|a|r}
\]
which is bigger than or equal to 1 if and only if
\[
r \geq \frac{1}{1 + k + (2 + k)|a|} = \frac{K + 1}{2K + (3K + 1)|a|}.
\]
This shows that the number $(K + 1)/(5K + 1)$ cannot be improved, since $|a|$ could be chosen so close to $1^-$. \qed

Proof of Theorem 1.2. Just adopt the method of proof of Theorem 1.1. The desired conclusion follows if we consider (2.1) and replace the first term $|a_0|$ in (2.1) by $|a_0|^2$. So we omit the details. \qed

Proof of Theorem 1.3. We recall that if $p(z) = \sum_{n=1}^{\infty} p_n z^n$ is analytic in $\mathbb{D}$ such that $\text{Re} \ p(z) > 0$ in $\mathbb{D}$, then $|p_n| \leq 2\text{Re} \ p_0$ for all $n \geq 1$. Applying this result to $p(z) = 1 - f(z)$ leads to $|a_n| \leq 2(1 - a_0)$ for all $n \geq 1$. Thus, as in the proof of Theorem 1.1, we can easily obtain from Lemma 2.1 that
\[
\sum_{n=1}^{\infty} |b_n|^2 r^n \leq k^2 \sum_{n=1}^{\infty} |a_n|^2 r^n \leq 4k^2(1 - a_0)^2 \frac{r}{1-r}
\]
and
\[
\sum_{n=1}^{\infty} |b_n| r^n \leq 2k(1 - a_0) \frac{r}{1-r}
\]
so that
\[
\sum_{n=0}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \leq a_0 + 2(1 + k)(1 - a_0) \frac{r}{1-r}
\]
which is clearly less than or equal to 1 for $r \leq 1/(3 + 2k)$. Again, substituting $k = (K - 1)/(K + 1)$ gives the desired result. Moreover, sharpness can be seen by considering functions of the form
\[
h(z) = \frac{a - z}{1 - \overline{a}z}, \quad 0 < a < 1, \quad \text{and} \quad g(z) = kh(z) = \frac{K - 1}{K + 1}h(z).
\]
The proof is complete. \qed
For the proof of Theorem 1.6, without loss of generality, we may assume that
\[ \max \{ \| h \|_\infty , \| g \|_\infty \} = 1. \]
Then, \( \| h \|_\infty \leq 1 \) and it follows from the classical Schwarz inequality that
\[
\sum_{n=1}^{\infty} |a_n|r^n \leq \sqrt{\sum_{n=1}^{\infty} |a_n|^2} \sqrt{\sum_{n=1}^{\infty} r^{2n}} \leq \frac{r}{\sqrt{1-r^2}}.
\]
Again, since \( \| g \|_\infty \leq 1 \), the same inequality is valid for \( b_n \). Thus, by combining the resulting inequality with the last inequality, we find that
\[
\sum_{n=1}^{\infty} (|a_n| + |b_n|)r^n \leq \frac{2r}{\sqrt{1-r^2}} \leq 1 \quad \text{for} \quad r \leq \frac{1}{\sqrt{5}}.
\]
Although this simple approach gives a good estimate, the number \( 1/\sqrt{5} \) is not sharp. In order to obtain the sharp estimate we will use a recent approach of Kayumov and Ponnusamy [11,12] which, in particular, settled the problem Ali et al. [2] on the Bohr radius for odd analytic functions.

**Lemma 2.2.** Suppose \( p \) is a natural number and \( 2r^{2p} < 1 \). If \( h(z) = \sum_{n=0}^{\infty} a_{pn+1} z^{pn+1} \) is analytic and \( |h(z)| \leq 1 \) for \( z \in \mathbb{D} \), then the following inequalities hold:
\[
\sum_{n=0}^{\infty} |a_{pn+1}|r^{pn+1} \leq \begin{cases} 
\frac{1}{r^{p-1}} \left( 3 - 2\sqrt{2} \sqrt{1 - r^2} \right) & \text{for} \quad |a_1| \geq r^p, \\
2r^{p+1} & \text{for} \quad |a_1| < r^p.
\end{cases}
\] (2.3)
If \( r^p \leq 1/3 \), then always
\[
\sum_{n=0}^{\infty} |a_{pn+1}|r^{pn+1} \leq \max \{ 2r^{p+1}, r \}
\]
holds.

**Proof.** This is a special case of the proof of Theorem 1 in [11,12]. So, we omit the details to avoid repetition. \( \square \)

**Proof of Theorem 1.6.** Since \( 3 - 2\sqrt{2} \sqrt{1 - r^2} = 1/2 \) gives \( r = \sqrt{7/32} \) and \( 2r^2 = 14/32 < 1/2, \) it follows from Lemma 2.2 that
\[
\sum_{n=0}^{\infty} |a_{n+1}|r^{n+1} \leq \frac{1}{2} \quad \text{for} \quad r \leq \sqrt{\frac{7}{32}}
\]
and the same inequality is valid for the coefficients \( b_n \):
\[
\sum_{n=0}^{\infty} |b_{n+1}|r^{n+1} \leq \frac{1}{2} \quad \text{for} \quad r \leq \sqrt{\frac{7}{32}}.
\]
Summing these two inequalities immediately proves that
\[
\sum_{n=1}^{\infty} (|a_n| + |b_n|)r^n \leq 1 \quad \text{for} \quad r \leq \sqrt{\frac{7}{32}}.
\]
To prove that the number \( \sqrt{7/32} \) cannot be replaced by a larger one, we let \( a = 3/\sqrt{14}, \lambda \in \partial \mathbb{D} \) and consider
\[
h(z) = z \left( \frac{a - z}{1 - az} \right) = az - (1 - a^2) z^2 \sum_{n=2}^{\infty} (az)^{n-2} \quad \text{and} \quad g(z) = \lambda h(z).
\]
Then, we find that
\[ \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n = 2 \left( ar + \frac{(1 - a^2) r^2}{1 - ar} \right) = 1 \text{ for } r = \sqrt{\frac{7}{32}} \]

which shows that the number \( \sqrt{7/32} \) cannot be improved. \( \square \)

**Proof of Theorem 1.7.** Let \( r = 1/2 \). Since \( r^p < 1/3 \) for \( p \geq 2 \), it follows from Lemma 2.2 and the hypothesis that
\[ \sum_{n=0}^{\infty} |a_{pn+1}| r^{pn+1} \leq \frac{1}{2} \text{ for } r \leq \frac{1}{2} \]

and the same inequality is valid for the coefficients \( b_n \). As a consequence of adding these two inequalities, we obtain that
\[ \sum_{n=0}^{\infty} (|a_{pn+1}| + |b_{pn+1}|) r^{pn+1} \leq 1 \text{ for } r \leq \frac{1}{2} \]

The function \( f(z) = z + \bar{z} \) shows that \( 1/2 \) is sharp. \( \square \)

### 3 | FURTHER RESULTS ON BOHR RADIUS FOR QUASICONFORMAL HARMONIC MAPPINGS

**Theorem 3.1.** Suppose that \( f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=2}^{\infty} b_n z^n \) is a sense-preserving \( K \)-quasiconformal harmonic mapping of the disk \( \mathbb{D} \), where \( h \) is a bounded function in \( \mathbb{D} \). Then
\[ \sum_{n=0}^{\infty} |a_n| r^n + \sum_{n=2}^{\infty} |b_n| r^n \leq ||h||_{\infty} \text{ for } r \leq r_K \]

where \( r_K \) is the positive root of the equation \( M_K(r) = 1/2 \) and
\[ M_K(r) = \frac{r}{1-r} + \left( \frac{K-1}{K+1} \right) r^2 \sqrt{\frac{1 + r^2}{(1 - r^2)^3}} \sqrt{\frac{\pi^2}{6} - 1} \]

The number \( r_K \) cannot be replaced by a number greater than \( R = R(K) \), where \( R \) is the positive root of the equation
\[ \frac{4R}{1-R} \left( \frac{K}{K+1} \right) + 2 \left( \frac{K-1}{K+1} \right) \log(1-R) = 1 \]

**Proof.** Without loss of generality we may assume that \( ||h||_{\infty} \leq 1 \). Because \( f = h + g \) is sense-preserving and \( K \)-quasiconformal harmonic mapping with \( g'(0) = 0 \), Schwarz's lemma gives that \( \omega f = \omega = g' / h' \) is analytic in \( \mathbb{D} \) and \( |\omega(z)| \leq k|z| \) in \( \mathbb{D} \), where \( k = (K-1)/(K+1) \). Thus, we have
\[ |g'(z)|^2 = |\omega(z)h'(z)|^2 \leq k^2 |zh'(z)|^2 \]

We integrate this inequality over the circle \( |z| = r \) and obtain
\[ \sum_{n=2}^{\infty} n^2 |b_n|^2 r^{2n-1} \leq k^2 r^2 \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-1} \]

and, because \( |a_n| \leq 1 - |a_0|^2 \) for \( n \geq 1 \), we have
\[ \sum_{n=2}^{\infty} n^2 |b_n|^2 r^{2n} \leq k^2 (1 - |a_0|^2)^2 r^3 \sum_{n=1}^{\infty} n^2 r^{2n-1} = k^2 (1 - |a_0|^2)^2 \frac{r^4 (1 + r^2)}{(1 - r^2)} \]
Consequently, using the classical Schwarz inequality, we deduce that
\[
\sum_{n=2}^{\infty} |b_n| r^n \leq \sqrt{\sum_{n=2}^{\infty} n^2 |a_n|^2 r^{2n}} \sqrt{\sum_{n=2}^{\infty} n^2} \leq kr^2 (1 - |a_0|^2) \sqrt{\frac{1 + r^2}{(1 - r^2)^3}} \sqrt{\frac{\pi^2}{6} - 1}. \tag{3.4}
\]
Therefore, from (3.4) it follows that
\[
S = |a_0| + |a_1| r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n \leq |a_0| + (1 - |a_0|^2) \left[ \frac{r}{1 - r} + kr^2 \sqrt{\frac{1 + r^2}{(1 - r^2)^3}} \sqrt{\frac{\pi^2}{6} - 1} \right]
\]
which is less than or equal to 1 provided $M_K(r) = 1/2$ holds, where $M_K(r)$ is defined by (3.2).

Finally, we let $a \in [0, 1)$ and consider the functions
\[
h(z) = \frac{a - z}{1 - az} \quad \text{and} \quad g'(z) = kzh'(z).
\]
From here we find that
\[
|a_n| = a^{n-1} (1 - a^2) \quad \text{and} \quad |b_n| = k\left(\frac{n-1}{n}\right) a^{n-2} (1 - a^2), \quad n \geq 2,
\]
so that
\[
\sum_{n=0}^{\infty} |a_n| r^n + \sum_{n=2}^{\infty} |b_n| r^n = a + (1 - a^2) \sum_{n=1}^{\infty} a^{n-1} r^n + k \left(1 - a^2\right) \sum_{n=2}^{\infty} \frac{n-1}{n} a^{n-2} r^n
\]
\[
= a + (1 - a^2) \frac{r}{1 - ar} + k \left(1 - a^2\right) \frac{ar + (1 - ar) \log(1 - ar)}{a^2(1 - ar)}
\]
\[
= a + (1 - a^2) \left[ \frac{(a + k)r}{ar(1 - ar)} + k \frac{\log(1 - ar)}{a^2} \right],
\]
where $k = (K - 1)/(K + 1)$. Simple analysis shows that the last expression is less than or equal to 1 for all $a \in [0, 1)$ only in the case when $r \leq R = R(K)$ which is the positive root of Equation (3.3).

Allowing $K \to \infty$ in Theorem 3.1 shows that the root $r_\infty$ of the limiting case $M_\infty(r) = 1/2$, i.e.,
\[
\frac{r}{1 - r} + r^2 \sqrt{\frac{1 + r^2}{(1 - r^2)^3}} \sqrt{\frac{\pi^2}{6} - 1} = \frac{1}{2},
\]
gives the value $0.2942...$. We may now formulate this discussion as follows.

**Corollary 3.2.** Suppose that $f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=2}^{\infty} b_n z^n$ is a sense-preserving harmonic mapping of the disk $\mathbb{D}$, where $h$ is a bounded function in $\mathbb{D}$. Then
\[
\sum_{n=0}^{\infty} |a_n| r^n + \sum_{n=2}^{\infty} |b_n| r^n \leq ||h||_\infty \quad \text{for} \quad r \leq 0.2942... \tag{3.5}
\]
The number $0.2942...$ cannot be replaced by a number greater than $R = 0.299825...$, where $R$ is the positive root of the equation
\[
\frac{4R}{1 - R} + 2 \log(1 - R) = 1.
\]

Further remarks would be useful.

**Remark 3.3.** Also, it is worth pointing out that if the first term $|a_0|$ in (3.1) is replaced by $|a_0|^2$, then the conclusion of Corollary 3.2 takes the form
\[
|a_0|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=2}^{\infty} |b_n| r^n \leq ||h||_\infty \quad \text{for} \quad r \leq r_K
\]
where \( r_K \) is the positive root of the equation \( M_K(r) = 1 \) and \( M_K(r) \) is defined by (3.2). The number \( r_K \) cannot be replaced by a number greater than \( R = R(K) \), where \( R \) is the positive root of the equation

\[
\frac{2R}{1 - R} \left( \frac{K}{K + 1} \right) + \left( \frac{K - 1}{K + 1} \right) \log(1 - R) = 1.
\]

Again, the case \( K \to \infty \) (i.e. when the dilatation has the property that \(|\omega(z)| < 1 \) in \( \mathbb{D} \)) needs a special mention, since the corresponding value of \( r_\infty \) is 0.435668..., where the number 0.435668... is the root of the equation

\[
M_\infty(r) = 1,
\]

i.e.,

\[
\frac{r}{1 - r} + r^2 \sqrt{\frac{1 + r^2}{(1 - r^2)^3}} \sqrt{\frac{r^2}{6} - 1} = 1.
\]

Furthermore, the number 0.435668... cannot be replaced by a number greater than \( R(\infty) = 0.44182 \ldots \), where \( R(\infty) \) is the positive root of the equation

\[
2R^2 + \log(1 - R) = 1.
\]

**Theorem 3.4.** Suppose that \( f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n \) is a locally univalent \( K \)-quasiconformal harmonic mapping of the disk \( \mathbb{D} \), where \( h' \) is a bounded function in \( \mathbb{D} \). Then

\[
\frac{K + 1}{2K} \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^{n-1} \leq ||h'||_\infty \text{ for } r \leq 1/3
\]

and the number 1/3 is sharp.

The proof of Theorem 3.4 easily follows if we use the above method and the classical proof of Bohr’s 1/3-Theorem. The case \( K \to \infty \) gives the corresponding result for sense-preserving harmonic mappings of the disk \( \mathbb{D} \).

### 4 BOHR RADIUS FOR HARMONIC BLOCH FUNCTIONS

A harmonic function \( f \) is called a harmonic Bloch function if and only if

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2) \Lambda_f(z) < +\infty,
\]

where \( \Lambda_f(z) = |f_\bar{z}(z)| + |f_z(z)| \). The space of all harmonic Bloch functions, denoted by the symbol \( B_H \), forms a complex Banach space with the norm \( || \cdot || \) given by (see [9])

\[
||f||_{B_H} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) \Lambda_f(z),
\]

where \( \Lambda_f(z) = |f_\bar{z}(z)| + |f_z(z)| \). This is referred to as the harmonic Bloch norm and the elements of the harmonic Bloch space are called harmonic Bloch functions. Recently, the space \( B_H \) together with its various generalizations have been studied extensively. See for example, see [6–8].

Clearly, this definition coincides with the classical Bloch space \( B \) when \( f \) is analytic in \( \mathbb{D} \). We refer to the basic paper on this topic by Anderson et al. [3] and the book of Pommerenke [15].

**Theorem 4.1.** Let \( f \in B \) and \( ||f||_B \leq 1 \). Then

\[
\sum_{n=0}^{\infty} |a_n| r^n \leq 1 \text{ for } r \leq R = 0.55356 \ldots,
\]

where \( R \) is the positive solution to the equation

\[
1 - R + R \log(1 - R) = 0.
\]

The number \( R \) cannot be replaced by a number greater than 0.624162...
Proof. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be analytic and

\[
||f||_B = ||f(0)|| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| \leq 1.
\]

Thus, we have

\[
|f'(z)|^2 \leq \frac{(1 - |a_0|^2)}{(1 - |z|^2)^2}.
\]

We integrate this inequality over the circle \(|z| = r\) and obtain

\[
\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n} \leq r^2 \frac{(1 - |a_0|^2)}{(1 - r^2)^2}
\]

so that

\[
\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{n-1} \leq \frac{(1 - |a_0|^2)}{(1 - r)^2}
\]

for \(r < 1\).

We may now integrate this with respect to \(r\) (with limit from 0 to \(r\)) and obtain

\[
\sum_{n=1}^{\infty} n |a_n|^2 r^{n-1} \leq \frac{(1 - |a_0|^2)}{1 - r}
\]

for \(r < 1\), which by integration with respect to \(r\) gives

\[
\sum_{n=1}^{\infty} |a_n|^2 r^n \leq (1 - |a_0|^2) \log\frac{1}{1 - r}
\]

for \(r < 1\).

Consequently,

\[
\sum_{n=1}^{\infty} |a_n| r^n \leq \left( \sum_{n=1}^{\infty} |a_n|^2 r^n \right) \left( \sum_{n=1}^{\infty} r^n \right) \leq (1 - |a_0|) \sqrt{\log\frac{1}{1 - r}} \sqrt{\frac{r}{1 - r}}.
\]

It means that

\[
\sum_{n=0}^{\infty} |a_n| r^n \leq |a_0| + (1 - |a_0|) \sqrt{\log\frac{1}{1 - r}} \sqrt{\frac{r}{1 - r}} \leq 1
\]

for \(r \leq R\), where \(R = 0.55356...\) is the positive solution to the equation

\[
\log\frac{1}{1 - R} = \frac{R}{1 - R}.
\]

Now we consider (see [4])

\[
f(z) = \frac{3\sqrt{3}}{4} \left( \left( \frac{z - a}{1 - az} \right)^2 - a^2 \right) = \frac{3\sqrt{3}}{4} \left( \sum_{n=1}^{\infty} a_n z^n \right),
\]

where \(a \in (0, 1)\), \(a_1 = -2a (1 - a^2)\) and

\[
a_n = (1 - a^2) \left( n (1 - a^2) - (1 + a^2) \right) (a)^{n-2}\text{ for } n \geq 2.
\]

Since

\[
f'(z) = \frac{3\sqrt{3}}{2} \left( \frac{z - a}{1 - az} \right) \frac{1 - a^2}{(1 - az)^2},
\]
it is easy to check that ||f||₁ = 1. For this function, we observe that the coefficients \( a_n \) for \( n \geq 2 \) are all positive, whenever \( a \in (0, 1/\sqrt{3}) \). Furthermore,

\[
\sum_{n=0}^{\infty} |a_n|r^n = \frac{3\sqrt{3}}{4} \left( \left( \frac{r-a}{1-ar} \right)^2 - a^2 + 4a(1-a^2)r \right). 
\]

Now we suppose that \( r \) is a function of \( a \). We want to find the minimal \( r \) for which

\[
\frac{3\sqrt{3}}{4} \left( \left( \frac{r-a}{1-ar} \right)^2 - a^2 + 4a(1-a^2)r \right) = 1
\]

which may be rewritten as

\[
(r-a)^2 + (1-ra)^2(-a^2 + 4a(1-a^2)r) = \frac{4}{3\sqrt{3}}(1-ar)^2. \quad (4.1)
\]

Now we differentiate \( r \) in the variable \( a \) and then we set \( r'(a) = 0 \). We arrive at the cubic equation

\[
18r + 8\sqrt{3}r - 54a^2r - 144ar^2 - 8\sqrt{3}ar^2 + 252a^3r^3 + 108a^2r^3 - 180a^4r^3 = 0. \quad (4.2)
\]

The algebraic system of Equations (4.1) and (4.2) can be easily solved, for example, by Mathematica 10. Consequently, we get \( a = 0.3775 \) and then we obtain \( r = 0.624162 \) such that \( \sum_{n=0}^{\infty} |a_n|r^n = 1 \). The proof is complete.

**Theorem 4.2.** Suppose that \( f = h + \overline{g} \) is harmonic in \( \mathbb{D} \), \( g(0) = 0 \) and \( ||f||_{B_H} \leq 1 \), where

\[
||f||_{B_H} = |f(0)| + \sup_{z \in \mathbb{D}} (1-|z|^2)(|h'(z)| + |g'(z)|).
\]

Then

\[
|a_0| + \sum_{n=1}^{\infty} \sqrt{|a_n|^2 + |b_n|^2}r^n \leq 1 \quad \text{for} \quad r \leq R = 0.55356.
\]

This number 0.55356 cannot be replaced by a number greater than 0.624162....

**Proof.** By assumption, we have

\[
|h'(z)|^2 + |g'(z)|^2 \leq \frac{(1-|a_0|)^2}{(1-|z|^2)}.
\]

We integrate this inequality over the circle \( |z| = r \) and obtain

\[
\sum_{n=1}^{\infty} n^2 (|a_n|^2 + |b_n|^2) r^{2n} \leq r^2 \frac{(1-|a_0|)^2}{(1-r^2)^2}.
\]

Now the remaining part of the proof is identical to that of Theorem 4.1. Thus, the proof is complete.

As remarked earlier if we replace the first term \( |a_0| \) in the conclusion of the last two theorems by \( |a_0|^2 \), then the Bohr radius obviously can be stated in an improved form.

**5 | CONCLUSION**

We conclude the paper with the following conjectures.
Conjecture 1. Suppose that \( f = h + g = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n \) is a harmonic mapping of the disk \( \mathbb{D} \), where \( h \) is bounded in \( \mathbb{D} \). If \( |g'(z)| \leq \frac{K-1}{K+1} |h'(z)| \), then
\[
\sum_{n=1}^{\infty} (|a_n| + |b_n|) r^n \leq ||h||_\infty \quad \text{for} \quad r \leq \frac{1}{4} \sqrt{\frac{2}{2 - \frac{1}{2K^2} + \frac{5}{K}}},
\]
This constant is sharp for all \( K \geq 1 \).

The conjectured extremal function has the form
\[
f(z) = h(z) + \frac{K-1}{K+1} \overline{h(z)}, \quad h(z) = z \left( \frac{z - a}{1 - \overline{a}z} \right),
\]
with a suitable \( a \in \mathbb{D} \).

If we replace the condition “\( |g'(z)| \leq \frac{K-1}{K+1} |h'(z)| \)” by “\( f \) is \( K \)-quasiconformal”, then the Bohr radius will be greater than the number mentioned in Conjecture 1, because the conjectured extremal function is not locally univalent in the unit disk.

The proof of Theorem 1.7 leads to another problem which we state it now as a conjecture.

Conjecture 2. Let \( p \geq 2 \). Suppose that \( f(z) = h(z) + a g(z) = \sum_{n=0}^{\infty} a_n z^{pn+1} + \sum_{n=0}^{\infty} b_n z^{pn+1} \) is a harmonic \( p \)-symmetric and sense-preserving mapping in \( \mathbb{D} \), where \( h \) and \( g \) are bounded functions in \( \mathbb{D} \). Then
\[
\sum_{n=0}^{\infty} (|a_n| + |b_n|) r^{pn+1} \leq ||h||_\infty, \quad \text{for} \quad r \leq \frac{1}{2}.
\]
The constant \( 1/2 \) is sharp.

Also it would be interesting to obtain an analog of the Conjecture 2 for locally \( K \)-quasiconformal mappings.

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