New predictor-corrector scheme for solving nonlinear differential equations with Caputo-Fabrizio operator

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MSC Classification: 34A08; 65D15; 65L99

1 INTRODUCTION

In 2015, Michele Caputo and Mauro Fabrizio introduced a new definition of Caputo derivative,¹ which was called Caputo-Fabrizio fractional derivative. The main characteristics of this new definition are including regular kernel, having two different representations for temporal and spatial variables and as memory operator. Although it was verified that the operator does not fit the usual concepts neither for fractional nor for integer derivative/integral,² the new operator was successfully applied in describing the control movement of waves on the area of shallow water,³ unsteady flows of an incompressible Maxwell fluid,⁴ anomalous diffusion phenomena,⁵ diffusion and the diffusion-advection equation,⁶ epidemiological model for computer viruses,⁷ model of circadian rhythms,⁸ and many more phenomena. In this research direction, some existing numerical methods have been extended to the problem defined in the sense of this new Caputo-Fabrizio operator. Among them is the combination of the Laplace transform and homotopy analysis method to solve the so-called fractional partial differential equation problem with Caputo-Fabrizio operator.⁹ Other methods include multiquadric (MQ)-RBF collocation method,¹⁰ discretization scheme,¹¹ fractional Adams-Bashforth method,¹² and three-step fractional Adams-Bashforth method.¹³ However, since Caputo-Fabrizio operator is relatively new, there are still relatively limited works that have been done to obtain the simple, reliable, and accurate solution for the problem defined in Caputo-Fabrizio operator.

On the other hand, predictor-corrector scheme had been extended to solve fractional differential equations.¹⁴ Since that, some modification was done to obtain a better accurate scheme or more suitable to different problems, such as
high-order predictor-corrector method for fractional differential equations,\cite{15} fractional Adams-Bashforth-Moulton methods for solving fractional Keller-Segel chemotaxis system,\cite{16} Adams-Bashforth-Moulton methods for solving fractional partial differential equations,\cite{17} and solving fractional predation system in subdiffusion and superdiffusion scenarios.\cite{18} Apart from that, some problems had been successfully solved by using the scheme developed by Diethelm,\cite{14} which includes fractional order delay-varying computer virus propagation model,\cite{19} nonlinear fractional partial differential equations.\cite{20} Here, we extend the use of predictor-corrector scheme or Adams-Bashforth-Moulton methods to solve the differential equation involving Caputo-Fabrizio operator.

The rest of the paper is organized as follows: Section 2 briefly explains Caputo-Fabrizio operator. Section 3 presents the predictor-corrector scheme suits on the Caputo-Fabrizio operator. Section 4 shows the errors for the quadrature formula, differential equation involving Caputo-Fabrizio operator.

In this section, we discuss the fractional difference method for fractional ordinary differential equation (FODE) involving Caputo-Fabrizio operator in Equation 3 for $\alpha > 0$ as follows:

$$c_F D^\alpha_{0,x} f(x) = g(x, f(x)), \quad x \in [0, X],$$

with the initial condition, $i = 0, 1, \ldots, n - 1$ where $n = \lfloor \alpha \rfloor$

$$f^{(i)}(0) = f^{(i)}_0.$$

We have the following Theorem.

**Theorem 1.** The initial value problem of Equation 3 is

$$f(x) = T_{n-1}(x) + \frac{(1 - \lfloor \alpha \rfloor)}{M(\lfloor \alpha \rfloor)(n - 2)!} \int_0^x (x - t)^{n-2} g(t, f(t))dt + \frac{\lfloor \alpha \rfloor}{M(\lfloor \alpha \rfloor)(n - 1)!} \int_0^x (x - t)^{n-1} g(t, f(t))dt,$$

where $n = \lfloor \alpha \rfloor$, decimal part, and integer part of $\alpha$, respectively.

**Definition 1.** For $a \in [0, 1], a \in (-\infty, x)$, and $f \in H^1(a, b)$, where $b > a$, the left-sided Caputo-Fabrizio operator is defined as follows:

$$c_F D^\alpha_{a,x} f(x) = M(\lfloor \alpha \rfloor) \left[ x \int_0^{x-a} f'(t)dt \right]$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$.

With the assumption of $f^{(s)}(a) = 0, s = 1, 2, \ldots, \lfloor \alpha \rfloor$, the general formula of Caputo-Fabrizio operator for $n - 1 < \alpha < n$, $\alpha \in \mathbb{R}^+$ can be obtained from Equation 1 as follows:

$$c_F D^\alpha_{a,x} f(x) = c_F D^\lfloor \alpha \rfloor_{a,x} (D^{\alpha} f(x))$$

$$= M(\lfloor \alpha \rfloor) \left[ x \int_0^{x-a} f'^{(\lfloor \alpha \rfloor+1)}(t)e^{- \frac{|x-t|}{1-\alpha}} dt \right]$$

$$f^{(\lfloor \alpha \rfloor+1)}(x) = D^{\lfloor \alpha \rfloor+1} f(x) = D^{\alpha} f(x),$$

where $[\alpha], \lfloor \alpha \rfloor, \lceil \alpha \rceil$, and $\lfloor \alpha \rfloor$ are the floor,$\alpha$, ceil,$\alpha$, decimal part, and integer part of $\alpha$, respectively.
where \( T_{n-1}(x) \) is the Taylor expansion of \( f(x) \) centered at \( x_0 = 0 \)

\[
T_{n-1}(x) = \sum_{i=0}^{n-1} \frac{x^i}{i!} f^{(i)}(0).
\]

**Proof.** The proof is following Section 4 in Loh et al.\(^{21}\) Taking Laplace transform of both sides of Equation 3, we have

\[
\mathcal{L}[\mathrm{CF}D_{0,x}^s f(x)] = \mathcal{L}[g(x)], \quad s > 0,
\]

\[
\frac{M(\{\alpha\})}{s(1 - \{\alpha\}) + \{\alpha\}} \left( s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0) \right) = G(s).
\]

Thus,

\[
F(s) = \frac{1}{s^n} \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0) + \frac{(1 - \{\alpha\})}{s^n M(\{\alpha\})} G(s) + \frac{\{\alpha\}}{s^n M(\{\alpha\})} G(s)
\]

\[
= \sum_{k=1}^{n} \frac{1}{s^n} s^{n-k} f^{(k-1)}(0) + \frac{(1 - \{\alpha\})}{M(\{\alpha\})} \left( \frac{1}{s^{n-1}} G(s) \right) + \frac{\{\alpha\}}{M(\{\alpha\})} \left( \frac{1}{s^n} G(s) \right).
\]

By applying inverse Laplace transform properties,

\[
f(x) = \sum_{i=0}^{n-1} \frac{x^i}{i!} f^{(i)}(0) + \frac{(1 - \{\alpha\})}{M(\{\alpha\})(n-2)!} \int_0^x (x - t)^{n-2} g(t) dt + \frac{\{\alpha\}}{M(\{\alpha\})(n-1)!} \int_0^x (x - t)^{n-1} g(t) dt.
\]

Hence, equivalently, we have

\[
f(x) = \sum_{i=0}^{n-1} \frac{x^i}{i!} f^{(i)}(0) + \frac{(1 - \{\alpha\})}{M(\{\alpha\})(n-2)!} \int_0^x (x - t)^{n-2} g(t) dt + \frac{\{\alpha\}}{M(\{\alpha\})(n-1)!} \int_0^x (x - t)^{n-1} g(t) dt.
\]

The proof of Theorem 1 is completed. \(\square\)

Besides that, the initial value problem in Equation 5 can be approximated and derived as Adams-Bashforth-Moulton method.

\[
f(x_{k+1}) = T_{n-1}(x_{k+1}) + \frac{(1 - \{\alpha\})}{M(\{\alpha\})(n-2)!} \int_0^{x_{k+1}} (x_{k+1} - t)^{n-2} u(t) dt + \frac{\{\alpha\}}{M(\{\alpha\})(n-1)!} \int_0^{x_{k+1}} (x_{k+1} - t)^{n-1} u(t) dt
\]

(7)

The integration part can be approximated by

\[
\int_0^{x_{k+1}} (x_{k+1} - t)^{n-2} u(t) dt \approx \int_0^{x_{k+1}} (x_{k+1} - t)^{n-2} \tilde{u}_{k+1}(t) dt,
\]

\[
\int_0^{x_{k+1}} (x_{k+1} - t)^{n-1} u(t) dt \approx \int_0^{x_{k+1}} (x_{k+1} - t)^{n-1} \tilde{u}_{k+1}(t) dt,
\]

(8)

where \( \tilde{u}_{k+1}(t) \) is the approximation of \( u(t) \).

By the determining \( \tilde{u}_{k+1}(t) \) with \( 0 \leq i \leq k \) or in other words, we instead integrate the part from Equation 8 by the rectangle rule, we obtain the explicit method, fractional Euler method. The approximation solution, \( f(x_i) \approx \tilde{f}_i \) at discrete space \( x_i \).

Thus, the predictor formula, \( f_{k+1}^p \), can be determined by the fractional Adams-Bashforth method as follows:

\[
f_{k+1}^p = T_{n-1}(x_{k+1}) + \frac{(1 - \{\alpha\})}{M(\{\alpha\})(n-2)!} \sum_{i=0}^{k} c_i^{k+1} g(x_i, f_i) + \frac{\{\alpha\}}{M(\{\alpha\})(n-1)!} \sum_{i=0}^{k} d_i^{k+1} g(x_i, f_i),
\]

(9)
where

\[
\begin{align*}
    c_{i,k+1} &= \frac{h^{n-1}}{n-1}[(k-i+1)^{n-1} - (k-i)^{n-1}], \\
    d_{i,k+1} &= \frac{h^n}{n}[(k-i+1)^n - (k-i)^n].
\end{align*}
\]

(10)

If \( \tilde{u}_{k+1}(s) \mid_{x=x_{k+1}} = \frac{x_{k+1}-x}{h} u(x) + \frac{x-x_{k+1}}{h} u(x_{k+1}) \), we obtain the implicit method, fractional trapezoidal rule. The corrector formula, \( f_{k+1} \), can be determined by the fractional Adams-Moulton method as follows:

\[
f_{k+1} = T_{n-1}(x_{k+1}) + \frac{(1 - \{\alpha\})}{M(\{\alpha\})(n-2)!} \left( \sum_{i=0}^{k} a_{i,k+1} g(x_i, f_i) + a_{k+1,k+1} g(x_{k+1}, f_{k+1}^p) \right)
\]

\[
+ \frac{\{\alpha\}}{M(\{\alpha\})(n-1)!} \left( \sum_{i=0}^{k} b_{i,k+1} g(x_i, f_i) + b_{k+1,k+1} g(x_{k+1}, f_{k+1}^p) \right),
\]

(11)

where

\[
a_{i,k+1} = \frac{h^{n-1}}{n(n-1)} \begin{cases} 
    k^n - (k+1)^{n-1}(k-n+1), & i = 0 \\
    (k-i+2)^n - 2(k-i+1)^n + (k-i)^n, & 1 \leq i \leq k, \\
    1, & i = k+1
\end{cases}
\]

\[
b_{i,k+1} = \frac{h^n}{n(n+1)} \begin{cases} 
    k^{n+1} - (k+1)^n(k-n), & i = 0 \\
    (k-i+2)^{n+1} - 2(k-i+1)^{n+1} + (k-i)^{n+1}, & 1 \leq i \leq k, \\
    1, & i = k+1
\end{cases}
\]

(12)

We obtain the predictor in Equation 9 by calculating an initial approximation \( f_{k+1}^P \) from current value \( f_k \). Meanwhile, the corrector in Equation 11 applies the approximation, \( f_{k+1}^P \), in order to obtain the refined corrector value of \( f_{k+1} \), which is used in next iteration successively.

## 4 Errors of Quadrature Formulas

In this section, we show some important information on errors of quadrature formulas of predictor and corrector step. The proving of Theorem is basically adapted from the paper by Kai Diethelm.\textsuperscript{22} Firstly, we present a statement for predictor by the rectangle rule.

**Theorem 2.** Let \( q(x) \in C^4[0,X] \); therefore, the error of approximation of Equation 5 by using the rectangle rule as in Equation 9 is given by

\[
\left| \int_0^{x_{k+1}} (x_{k+1} - x)^{n-2} q(x) \, dx - \sum_{i=0}^{k} c_{i,k+1} q(x_i) \right| + \left| \int_0^{x_{k+1}} (x_{k+1} - x)^{n-1} q(x) \, dx - \sum_{i=0}^{k} d_{i,k+1} q(x_i) \right| \leq h \| q' \|_\infty \left( \frac{x_{k+1}^{n-1}}{n-1} + \frac{x_{k+1}^n}{n} \right). 
\]

(13)

**Proof.** The quadrature error of predictor step can be expressed as in Equation 14 by the rectangle formula as follows:

\[
\left| \int_0^{x_{k+1}} (x_{k+1} - x)^{n-2} q(x) \, dx - \sum_{i=0}^{k} c_{i,k+1} q(x_i) \right| + \left| \int_0^{x_{k+1}} (x_{k+1} - x)^{n-1} q(x) \, dx - \sum_{i=0}^{k} d_{i,k+1} q(x_i) \right|
\]

\[
= \left| \sum_{i=0}^{k} \int_{ih}^{(i+1)h} (x_{k+1} - x)^{n-2} (q(x) - q(x_i)) \, dx \right| + \left| \sum_{i=0}^{k} \int_{ih}^{(i+1)h} (x_{k+1} - x)^{n-1} (q(x) - q(x_i)) \, dx \right|.
\]

(14)

Here, we use the mean value theorem to the right-hand side (RHS) of Equation 14 in order to prove Theorem 2 and its derivation as shown as follows:
\[
\left| \int_0^{x_{k+1}} (x_{k+1} - x)^{n-2} q(x) dx - \sum_{i=0}^k c_{i,k+1} q(x_i) \right| \leq \left| \int_0^{x_{k+1}} (x_{k+1} - x)^{n-1} q(x) dx - \sum_{i=0}^k d_{i,k+1} q(x_i) \right| 
\]

\[
\leq \|q\|_{\infty} \left( \sum_{i=0}^k \left( \frac{k+1}{n+1} \right)^n - \left( \frac{k-i}{n+1} \right)^n - \left( \frac{k}{n+1} \right)^n \right) 
\]

\[
= \|q\|_{\infty} \left( \frac{n}{n-1} + \frac{h^{n+1}}{n} \right) = h \|q\|_{\infty} \left( \frac{x^{n-1}}{n-1} + \frac{x^n}{n} \right). 
\]

\[
(16)
\]

There are two different parentheses, which are the remainder of rectangle quadrature formula with the integrand, \(x^{n-1}\) and \(x^n\). These two integrands, \(x^{n-1}\) and \(x^n\), are monotonic on the interval \([0, k+1]\). On the basis of monotonicity property of integrand from quadrature theory, the term in both parentheses is bounded by the total variation of the integrands, \((k + 1)^{n-1}\) and \((k + 1)^n\), respectively.

Hence,

\[
\left| \int_0^{x_{k+1}} (x_{k+1} - x)^{n-2} q(x) dx - \sum_{i=0}^k c_{i,k+1} q(x_i) \right| \leq h \|q\|_{\infty} \left( \frac{x^{n-1}}{n-1} + \frac{x^n}{n} \right). 
\]

\[
(17)
\]

The proof of Theorem 2 is completed.

**Theorem 3.** Let \(q(x) = x^p\) for some \(p \in (0, 1)\). Thus,

\[
\left| \int_0^{x_{k+1}} (x_{k+1} - x)^{n-2} q(x) dx - \sum_{i=0}^k c_{i,k+1} q(x_i) \right| \leq h \left( \frac{K_{n-1,p}^R}{n-1} x^{n+p-2} + \frac{K_{n,p}^R}{n} x^{n+p-1} \right), 
\]

where \(K_{n-1,p}^R\) and \(K_{n,p}^R\) are constant that depend on \(n\) and \(p\).

**Proof.** In similar way, by applying mean value theorem and monotonicity of \(p\), the proof of Theorem 3 is shown as follows:

\[
\left| \int_0^{x_{k+1}} (x_{k+1} - x)^{n-2} q(x) dx - \sum_{i=0}^k c_{i,k+1} q(x_i) \right| \leq h \left( \frac{K_{n-1,p}^R}{n-1} x^{n+p-2} + \frac{K_{n,p}^R}{n} x^{n+p-1} \right). 
\]

\[
(17)
\]
Equation 18 is bounded by Theorem 4.

where \( K_{Tr} \) with \( z \) by using the Euler-MacLaurin formula.

Next, we present a corresponding statement for corrector by the trapezoidal rule, which is similar to Theorem 2;

This completes the proof of Theorem 3.

Hence,

\[
\left( \sum_{i=0}^{k} (i+1)^p - (i)^p \right) [(k+1-i)^n - (k-i)^n] \leq \frac{h^{p+n}}{n-1} \left( 2(k+1)^n - 2k^n + p(n-1) \sum_{i=1}^{k-1} i^{p-1} (k-i+z)^{n-2} \right) + \frac{h^{p+n}}{n} \left( 2(k+1)^n - 2k^n + pn \sum_{i=1}^{k-1} i^{p-1} (k-i+z)^{n-1} \right)
\]

where \( z = 0 \) if \( n \leq 1 \) or \( z = 1 \) in otherwise. We generalize the term in these two parentheses from the last inequality of Equation 18 is bounded by \( K_{Re}^{n-1, p} (k+1)^{n+p-2} \) and \( K_{Re} (k+1)^{n+p-1} \), respectively, in sense of brief asymptotic analysis by using the Euler-MacLaurin formula.

Hence,

\[
\left\| \frac{x_{k+1}}{0} \int (x_{k+1} - x)^{n-2} q(x) dx - \sum_{i=0}^{k} c_{i,k+1} q(x_i) \right\| + \left\| \frac{x_{k+1}}{0} \int (x_{k+1} - x)^{n-1} q(x) dx - \sum_{i=0}^{k} b_{i,k+1} q(x_i) \right\| \leq h \left( K_{Re}^{n-1, p} x_{k+1}^{n+p-2} + K_{Re} x_{k+1}^{n+p-1} \right)
\]

where \( K_{Re}^{n-1, p} \) and \( K_{Re} \) are a constant depending on \( n \) and \( p \).

This completes the proof of Theorem 3.

Next, we present a corresponding statement for corrector by the trapezoidal rule, which is similar to Theorem 2; therefore, we omit the details.

**Theorem 4.** Let \( q \in C^2[0, X] \),

\[
\left( \sum_{i=0}^{k} (i+1)^p - (i)^p \right) [(k+1-i)^n - (k-i)^n] \leq \frac{h^{p+n}}{n-1} \left( 2(k+1)^n - 2k^n + p(n-1) \sum_{i=1}^{k-1} i^{p-1} (k-i+z)^{n-2} \right) + \frac{h^{p+n}}{n} \left( 2(k+1)^n - 2k^n + pn \sum_{i=1}^{k-1} i^{p-1} (k-i+z)^{n-1} \right)
\]

where \( K_{Re}^{n-1, p} \) and \( K_{Re} \) are a constant depending on \( n \).

5 | ERROR ANALYSIS FOR THE ADAMS-BASHFORTH-MOULTON METHOD

We present a general convergence result for the Adams method by extending error estimates of Section 4.

**Lemma 1.** Assume

\[
\left( \sum_{i=0}^{k} (i+1)^p - (i)^p \right) [(k+1-i)^n - (k-i)^n] \leq K_{1} x_{k+1}^{\gamma_1} h^{\delta_1} + K_{2} x_{k+1}^{\gamma_2} h^{\delta_2}
\]

and

\[
\left( \sum_{i=0}^{k} (i+1)^p - (i)^p \right) [(k+1-i)^n - (k-i)^n] \leq K_{3} x_{k+1}^{\gamma_3} h^{\delta_3} + K_{4} x_{k+1}^{\gamma_4} h^{\delta_4},
\]

with \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \geq 0 \), and \( \delta_1, \delta_2 > 0 \).
Hence, for $X > 0$ and $N = \left\lfloor \frac{X}{h} \right\rfloor$,

$$\max_{0 \leq i \leq N} |f(x_i) - f_i| = O(h^r), \quad r = \min(\delta_1 + n - 1, \delta_2). \quad (23)$$

**Proof.** We will prove that

$$|f(x_i) - f_i| \leq Kh^r, \quad (24)$$

for sufficiently small $h$, where $K$ is a positive real constant and $i \in 0, 1, \ldots N$. On the basis of mathematical induction, induction basis holds for $i = 0$ by the given initial condition. After that, we assume that Equation 24 holds for $i \in 0, 1, \ldots k$ where $k \leq N - 1$ and then using this assumption to prove the inequality is true for $i = k + 1$. We first consider the error estimate of the predictor $f_{k+1}^P$ as follows:

$$\left| f(x_{k+1}) - f_{k+1}^P \right| = \frac{(1 - \{a\})}{M(\{a\})(n-2)!} \left( \frac{x_{k+1} - x}{m} \right)^{n-2} \frac{g(x, f(x))}{dx} - \sum_{i=0}^{k} c_{l,k+1} g(x_i, f_i) \right) \right|$$

$$\leq \frac{(1 - \{a\})}{M(\{a\})(n-2)!} \left( \int_{x_{k+1}}^{x_n} (x_{k+1} - x)^{n-2} g(x, f(x))dx - \sum_{i=0}^{k} c_{l,k+1} g(x_i, f_i) \right)$$

$$\leq \frac{(1 - \{a\})}{M(\{a\})(n-2)!} \left( \sum_{i=0}^{k} c_{l,k+1} |g(x_i, f(x_i)) - g(x_i, f_i)| \right)$$

$$\leq \frac{(1 - \{a\})}{M(\{a\})(n-2)!} \sum_{i=0}^{k} c_{l,k+1} LK \int_{x_{k+1}}^{x_n} (x_{k+1} - x)^{n-2} g(x, f(x))dx$$

$$\leq \frac{(1 - \{a\})}{M(\{a\})(n-2)!} \sum_{i=0}^{k} c_{l,k+1} LK + \frac{(1 - \{a\})}{M(\{a\})(n-2)!} \sum_{i=0}^{k} d_{l,k+1} LK$$

The assumption on error of rectangle rule, Lipschitz property of $g$ in Equation 26 and quadrature formula of predictor in Equation 27, was used in this derivation of Equation 25.

The Lipschitz continuous function, $g$, has the existence and uniqueness of the initial value problem.

$$|g(x_i, f(x_i)) - g(x_i, f_i)| \leq L |f(x_i) - f_i| \quad (26)$$

The quadrature formula can be approximated by

$$\sum_{i=0}^{k} c_{l,k+1} = \int_{x_0}^{x_{k+1}} (x_{k+1} - x)^{n-2} dx = \frac{x_{k+1}^n - x_0^n}{n} \leq \frac{X^n}{n}$$

$$\sum_{i=0}^{k} d_{l,k+1} = \int_{x_0}^{x_{k+1}} (x_{k+1} - x)^{n-1} dx = \frac{x_{k+1}^n - x_0^n}{n} \leq \frac{X^n}{n}. \quad (27)$$
We derive corrector error in a similar way with the assistance of predictor error in Equation 25.

\[
|f(x_{k+1}) - f_k| = \left| \frac{1 - \{a\}}{M(\{a\})(n-2)!} \left( \int_{0}^{x_{k+1}} (x_{k+1} - x)^n g(x) \, dx - \sum_{i=0}^{k+1} a_{i,k+1} g(x_i, f_i) \right) \right|
\]

\[
+ \frac{\{a\}}{M(\{a\})(n-1)!} \left( \int_{0}^{x_{k+1}} (x_{k+1} - x)^{n-1} g(x, f(x)) \, dx - \sum_{i=0}^{k+1} b_{i,k+1} g(x_i, f_i) \right)
\]

\[
\leq \frac{1 - \{a\}}{M(\{a\})(n-2)!} \left( \int_{0}^{x_{k+1}} (x_{k+1} - x)^n g(x) \, dx - \sum_{i=0}^{k+1} a_{i,k+1} g(x_i, f_i) \right)
\]

\[
+ \frac{\{a\}}{M(\{a\})(n-1)!} \left( \int_{0}^{x_{k+1}} (x_{k+1} - x)^{n-1} g(x, f(x)) \, dx - \sum_{i=0}^{k+1} b_{i,k+1} g(x_i, f_i) \right)
\]

\[
\leq \frac{1 - \{a\}}{M(\{a\})(n-2)!} \left( \int_{0}^{x_{k+1}} (x_{k+1} - x)^n g(x) \, dx - \sum_{i=0}^{k+1} a_{i,k+1} g(x_i, f_i) \right)
\]

\[
+ \frac{\{a\}}{M(\{a\})(n-1)!} \left( \int_{0}^{x_{k+1}} (x_{k+1} - x)^{n-1} g(x, f(x)) \, dx - \sum_{i=0}^{k+1} b_{i,k+1} g(x_i, f_i) \right)
\]

\[
\leq \left\{ \frac{1 - \{a\}}{M(\{a\})(n-2)!} \left( \frac{K_1 x_{k+1} \gamma \delta_2}{(n-2)!} + \frac{L K X^n}{(n-1)!} \right) + \frac{\{a\}}{M(\{a\})(n-1)!} \left( \frac{K_1 x_{k+1} \gamma \delta_2}{(n-1)!} + \frac{L K X^n}{n!(n-1)!} \right) \right\} h^n
\]

where \( \delta_1 + n \leq r, \delta_2 \leq r, \) and \( \gamma_1, \gamma_2 > 0. \)

\[\Box\]

6 NUMERICAL EXAMPLES

We show some numerical examples for nonlinear differential equation of Caputo-Fabrizio operator by using our proposed numerical method, which is the predictor-corrector scheme involving Caputo-Fabrizio operator.
Example 1. We consider the following:

\[ c_{FD}^4 f(x) + f^2(x) = M(\alpha) \left( \frac{5}{8} e^{-3x} - \frac{5}{8} \frac{x^5}{2} \right) + x^4. \] (28)

Let \( M(\alpha) = 1 \), the exact solution is given by \( f(x) = x^5 \). With \( h = \frac{1}{100} \), the approximate solution and absolute error for the propose new predictor-corrector scheme involving Caputo-Fabrizio operator are shown in Table 1.

Example 2. Consider the following:

\[ c_{FD}^3 f(x) + f^2(x) = M(\alpha) \left( 288e^{-3x} + 40x^3 - 120x^2 + 288x - 288 \right) + (x^5 + 4x^3)^2. \] (29)

Let \( M(\alpha) = 1 \), the exact solution is given by \( f(x) = x^5 + 4x^3 \). With \( h = \frac{1}{100} \), the approximate solution and absolute error for the propose new predictor-corrector scheme involving Caputo-Fabrizio operator are shown in Table 2.

7 | CONCLUSION

In this paper, a new predictor-corrector scheme for solving nonlinear differential equation involving Caputo-Fabrizio operator was derived. Error analysis and numerical examples shown that the scheme is highly efficient. The scheme can be easily extended to employ and solve the system of nonlinear differential equations involving Caputo-Fabrizio operator. In the future, we hope that this can also extend the scheme to solve some other problems related to newly defined differential operators or other types of fractional calculus problems such as in the study of Owolabi, KM (2018a), Owolabi, KM.

**ACKNOWLEDGEMENTS**

Financial supports from Universiti Tun Hussein Onn Malaysia in the form of TIER grant Vot H229 are gratefully acknowledged.

**CONFLICT OF INTEREST**

The authors declare that there is no conflict of interest regarding the publication of this manuscript.

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