A Nonlinear Deterministic Mode Decomposition Strategy for High-Dimensional Monte Carlo Simulations

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We present a new reduced Monte Carlo simulation strategy for nonlinear high-order systems based on an extension of the proper orthogonal decomposition for transient excited structures. This novel approach enables the evaluation of the response statistics in a fractional amount of time compared to the full Monte Carlo simulation procedure.

1 Introduction

Generally, Monte Carlo (MC) simulations of high-dimensional nonlinear systems are practically unrealizable. Therefore, equivalent linearization techniques [1] or weighted simulation methods [2] are proposed to handle this issue. In this paper, we present a reduced MC simulation strategy for high-order systems that is executed in a nonlinear reduced subspace. The main goal is to identify one representative transformation matrix that is applied to all sample computations within the MC simulation run. Thus, we propose an a-priori response identification by computation of the response of the structure excited by one representative sample event in the physical coordinate, that is to say an evaluation of representative snapshots. The proper orthogonal decomposition (POD) (e.g. [3–5]) is applied to evaluate the deterministic transformation matrix (POD modes) into one representative sample event in the physical coordinate, that is to say an evaluation of representative snapshots. The proper orthogonal decomposition (POD) (e.g. [3–5]) is applied to evaluate the deterministic transformation matrix (POD modes) into one representative sample event in the physical coordinate, that is to say an evaluation of representative snapshots.

2 Low-order representations of the nonlinear equation of motion

The set of equations of motion with \( n \) degrees of freedom for a transient ground accelerated structure is

\[
M \ddot{x} + C \dot{x} + R(x) = -Mf \ddot{x}_g^{(i)} \quad \rightarrow \quad D(\ddot{x}, \dot{x}, x) = F^{(i)}(t),
\]

where \( M \) and \( C \) are \( n \times n \) mass and stiffness matrices and \( R(x) \) is the \( n \times 1 \) nonlinear resisting force vector. The utilization of numeric integration schemes is essential in order to obtain the response function \( \mathbf{X}^{(i)}(t) \). Unfortunately, if \( n \) is a large number, the utilization of a numeric integration algorithm leads to high computational costs. Especially, the application of explicit integrators is considerably time-consuming as the algorithm becomes unstable if the chosen time step \( \Delta t \) exceeds a critical value \( \Delta t_{crit} [6] \). Consequently, a large number of loops has to be processed during the integration procedure. Therefore, a transformation matrix \( \mathbf{B} \) is defined that provides the transformation into the \( m \)-dimensional \((m \ll n)\) subspace in the reduced coordinate \( \mathbf{q}, \mathbf{x} = \mathbf{Bq} \), so that the response function of equation (1) is approximated as accurately as possible. The reduced set of equations of motion in the coordinate \( \mathbf{q}(t) \) then reads:

\[
\mathbf{B}^T \mathbf{M} \ddot{\mathbf{q}} + \mathbf{B}^T \mathbf{C} \dot{\mathbf{q}} + \mathbf{B}^T \mathbf{R} \mathbf{B} \mathbf{q} = -\mathbf{B}^T M \ddot{x}_g^{(i)} \quad \rightarrow \quad \mathbf{D}_B(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}) = \mathbf{B}^T F^{(i)}(t).
\]

Explicit numeric integration of the low-order system (2) leads to a fractional amount of computational time induced by the considerably larger critical time step.

The sample excitation term on the right hand side of equation (1), dependent on \( \ddot{x}_g^{(i)} \), i.e. the \( i^{th} \) ground acceleration sample, is realized by a stationary filtered white noise. The Kanai-Tajimi filter [2] \( S_{bb}(\omega) \) is applied to define the properties of an artificial earthquake excitation. Additionally, the time-dependent intensity is described by the envelope function \( e(t) \) [2]:
3 The reduced MC simulation strategy

According to the generated excitation sample set $\tilde{x}_g(t)$, the sample response set $x^{(1)}(t)$ must be evaluated. An a priori response identification $x^{(1)}(t)$, i.e. an observation function, is evaluated by solving the set of equations of motion, excited by the ground acceleration sample $\tilde{x}_g(t)$, in the physical coordinate:

$$\mathbf{M}\ddot{x}^{(1)} + \mathbf{C}\dot{x}^{(1)} + \mathbf{R}(x)^{(1)} = -\mathbf{Mf}_g(t) \quad \rightarrow \quad x^{(1)}(t) . \quad (4)$$

Next, the observation matrix $\mathbf{X} = [x^{(1)}(t_1), x^{(1)}(t_2), \ldots, x^{(1)}(t_s)]$ is evaluated by defining $s$ snapshots at different time instants within the observation time period $[t_0, t_{red}]$. In this step, it is of utmost important to capture main representative dynamic linear and nonlinear deformation patterns, otherwise the representative low-order approximations within the MC sampling computations are impossible. The left singular vectors of the snapshot matrix are defined as the POD modes and the corresponding singular values in descending order as the POD values. In order to define the deterministic transformation matrix $\mathbf{B}$, a truncation of the evaluated POD modes is done, so that 99.99 percent of the deformation energy within the snapshot matrix is captured. Following, the differential operator $\mathbf{D_B}(\dot{\mathbf{q}}, \mathbf{q})$ is defined based on the transformation matrix $\mathbf{B} = \mathbf{B}(\mathbf{D}, \tilde{x}_g^{(1)})$ according to transformation procedure (2). Then, the reduced response functions $\mathbf{q}^{(i)}(t)$ ($i = 1 \ldots m$), i.e. all MC simulation samples, are evaluated in this reduced subspace. Finally, the set of responses $\mathbf{q}^{(i)}(t)$ is transformed back into the physical space to obtain the set of approximated response functions $x^{(i)}_{red}(t)$:

$$\mathbf{D_B}(\dot{\mathbf{q}}, \mathbf{q}) = -\mathbf{B}^T \mathbf{Mf}_g \quad \rightarrow \quad < \mathbf{q}^{(1)}(t), \mathbf{q}^{(2)}(t), \ldots, \mathbf{q}^{(m)}(t) > \rightarrow < \mathbf{x}^{(1)}_{red}(t), \mathbf{x}^{(2)}_{red}(t), \ldots, \mathbf{x}^{(m)}_{red}(t) > . \quad (5)$$

In sum, the salient feature of this new strategy lies in the fact that only one transformation matrix is applied in order to obtain low-order representations within the whole MC sample computation run.

The numerical demonstration is presented on a ground accelerated two-story frame test structure discretized by finite beam elements (86 DOF, Fig. 1). Nonlinear elasto-plastic material behavior is implemented by a bilinear stress-strain relation with kinematic hardening (initial stiffness $E_0 = 2.1 \times 10^3 \ [N/m^2]$, post-yielding stiffness $E_1 = 2.1 \times 10^4 \ [N/m^2]$). The result of the POD-reduced, the full and the linear (initial stiffness) MC simulation is presented in Fig. 2 by the histogram of the absolute maximum displacement $h(t)$ shown in Fig. 1.

![Fig. 1: Two-story frame system, units in meter](image)

![Fig. 2: Response statistics: histogram, $10^3$ samples](image)

4 Conclusion

We propose a reduced-order MC simulation strategy based on the proper orthogonal decomposition method. One transformation matrix based on a priori response information is utilized to create a low-order representation of each sample computation. Accurate approximations of the response statistics compared to the full system are observed. However, it is of utmost importance to capture the main dynamic behavior within the a priori response information matrix (snapshot matrix) in order to ensure significant low-order representations with adequate accuracy.

References