Problems relating to the optimal design of transportation networks provide a challenge at a time when the transportation media are playing an increasingly important role in the day-to-day functioning of the space-economy. Transportation and communication networks are prominent features of both urban and rural landscapes. They transport persons, organic and inorganic matter in both solid and fluid states, gas, electricity, and information. Decisions to extend, locate or relocate these networks are frequently made at local, regional and national levels. The impacts of such decisions are widespread and frequently imply significant spatial reorganizations, which are especially critical in developing nations where budget constraints and high opportunity costs are prime factors in spatial planning decisions. The need to establish guidelines for such decisions is self-evident. With this in mind, an attempt is made in this paper to demonstrate within an exploratory context a theoretical framework for determining the “optimal network” design for a given, limited area.

In geographic terms, network designs are defined by their topology and/or geometric structures. For a network with a fixed topologic framework, the geometric structure can vary significantly, while the reverse is not true, as is illustrated by Figure 1. Clearly the geometry is crucial to a planned transportation network since it gives the network design its final fixed form. Therefore this study is designed to evaluate the optimal geometry of a network. In general, optimality is defined arbitrarily, depending on the goals of the study and the variables employed. The term “optimal network” as used here refers to the minimum total flow cost network for the given area, where traffic flows may originate from any point within the area.

Within this general class of problems, a limited city-hinterland case has been selected for analysis. Essential components of this regional level analysis comprise a well-defined rectangular area, a focal point located centrally at one
edge of this rectangle, and a trunk route passing through the point and bisecting the rectangular hinterland. Despite its seeming simplicity, this case is really quite complex mathematically, and it contains enough elements to clearly demonstrate the problems encountered in this type of analysis.

While certain aspects of transportation geography have been intensely investigated in the past, optimality of network geometry has received relatively little attention. Hitherto the role of geometry and explicitly the areal dimension have been assiduously omitted. In part this can be attributed to the simplistic a priori concepts necessary for the establishment of a simple model. In addition, combinatorial problems, the derivation of feasible solutions and the applicability of any such solution to the real world, serve to constrain the number of variables that can be included. The divergence of research on transportation networks is noticeable and its significance is manifold. To date, the bulk of the research has evolved around principles of optimal location and topological linkage of activities. In addition, attention has been focused at aggregate levels where considerations of geometry and area could be readily excluded. Methods of analysis have varied from simple graphic approaches to optimization techniques with emphasis on capacity constraints, mode, and problems of modal interchange.

One of the earliest illustrations of an awareness of topology was found in Christaller's communications principle and the associated network structure, which provided a deterministic model at the macrolevel. Beckmann [1] and Freidrich [4] both employed calculus of variations. The former considered the efficient allocation of transportation assuming a continuum of activities and a continuous cover by the transportation network. The latter integrated economic and cultural activities into an urban transportation network model with specified planning goals. Topologic concepts were emphasized by Garrison and Marble [7] in an attempt to examine the relationship between network structure and degree of economic development, utilizing simple concepts from graph theory. Linear programming techniques were utilized to introduce capacity constraints in another model proposed by Garrison and Marble [6]. Quandt's [9] approach is similar, but it emphasized the implications of a budget constraint on the capacity and construction costs of a planned network.

A methodology combining both topology and geometry has been utilized by only a few researchers in transportation. In geography Werner [15] has worked extensively on network geometry and optimal network designs. In
his discussion of the influence of topology and geometry on optimal network design, he notes that areal aggregation of flows along network lines is used to adjust spatial design to a minimum cost network configuration. However, the numerous alternative topological designs do restrict unique solutions to a few simple cases. Relevant research in other disciplines is best illustrated by the work of a number of geologists. Horton’s [8] analysis of stream patterns and drainage basins illustrates the high degree of correlation between natural and artificial transportation systems that is obtained by using the geometrical approach. This study was subsequently expanded by Strahler [12] and is further developed within a topological framework by Shreve [10].

The major limitation of the geometrical approach is that even in a limited case, combinatorial complexities increase rapidly with the addition of variables. However, this method does represent a step towards providing a basis for rational decision-making. It embodies in it the importance of relative location, a necessary aspect of planning. Therefore, it is hoped that the numerous gross errors in the planning field have given rise to sufficient consternation and interest to give impetus to the study of the basic problems of optimal network configuration.

**Model Construction**

The problem as defined is to establish the optimal network configuration, which connects a node with a portion of its hinterland by the addition of secondary routes to the trunk route, while minimizing the total movement costs from all points in the hinterland to the node. However a number of geometrically dissimilar and/or topologically similar patterns may be developed within this general organizing principle (Figure 2). Figure 2(b) was selected as the basic design for the study, since similar patterns are frequently observable in the real world at both regional and local levels.

To facilitate the construction of a model, certain basic parameters are defined as follows (Figure 3):

1. A homogeneous rectangle plane with areal dimensions, $a \cdot 2b$.
2. A trunk line from the node bisecting the hinterland.
3. Feeder lines linked to the trunk line at an angle $\alpha$.
4. Regular distribution of feeder lines at unit intervals $d$.
5. Differential movement costs of vehicles off and on the network, with $c_1$ designated as unit cost of travel off the network and $c_2$ designated as unit cost of travel on the network.

Two additional assumptions have also been built into the framework; first, $c_1 > c_2$ and secondly, that vehicular flow generation is uniformly distributed over the whole plane such that there is a demand for travel or that a vehicular movement can be generated from any point on the plane.

An arbitrary decision was made to optimize the network configuration with respect to total cost of movement, an aspect of decision-making elaborated
FIG. 2. Some Alternative Ways to Link a Node to Its Hinterland.

FIG. 3. General Network Configuration.

upon by Fromm [6]. This is, by definition, a function of budget (construction costs), length of route, and the magnitude of flow on the route. Construction cost in turn is dependent on construction time, budget, and the length of route. So a number of basic economic variables are implicitly present in the model.

Basic to minimization of total flow cost is the notion that movement from any point to the node will occur along a minimum flow cost path. To derive the minimum paths, use is made of the concept of "optimum slope." Stackelberg [II] and Beckmann [2] define the "optimum slope" as the "law of refraction of traffic," fulfilling the condition that total transportation cost between origin and destination be minimized for any flow using the trunk. The above term of reference is given to the preferred angle at which the minimum flow cost path connects any point to the nearest branch of the network by a straight line (Figure 4).
Thus, the optimum slope is defined as a functional relation between $c_1$ and $c_2$ by the following expression:

$$\text{Optimum slope} = \tan \theta = m = \sqrt{c_1^2 - c_2^2}/c_2$$

The study area can be represented by two halves, one a mirror image of the other, separated by the trunk route which is defined as the abscissa of the arbitrary grid pattern. All further reference will be made solely to the upper half, since each is a mirror image of the other.

As a preliminary step, a cartesian coordinate system was superimposed over the area such that the node of the focus was located at $(0, 0)$, the trunk line was represented by $y = 0$, and the fixed edge was $x = 0$. Each areal unit between two adjacent secondary routes will henceforth be known as a sector.

The diagrammatic representation of the given theoretical network in Figure 5 indicates a finite set of alternative minimum paths available for movements from each sector to the focus. For example, Sector I has two such paths and Sector II has four. The paths are derived as functions of $a, m, d$ and $b$, and are tabulated in Table 1.

A boundary of indifference exists between any two adjacent paths when the flow costs from a typical point $(a, b)$ to the node $(0, 0)$ are equal both ways. These boundaries between each direction of movement are obtained by equating the cost function of two adjacent paths (Table 2).

Representative equations for these boundaries were evaluated at the arbi-
TABLE 1

GENERAL MINIMUM COST FUNCTIONS OF MINIMUM FLOW COST PATHS

Sector I
1. $c_1 \sqrt{a^2 + b^4}$
2. $c_2(a \cos \alpha + b \sin \alpha) + (b \cos \alpha - a \sin \alpha)\sqrt{c_1^2 - c_2^2}$

Sector II
1. $c_2(a \cos \alpha + b \sin \alpha) - (b \cos \alpha - a \sin \alpha)\sqrt{c_1^2 - c_2^2}$
2. $c_1 \sqrt{a^2 + b^4}$
3. $ac_2 + b\sqrt{c_1^2 - c_2^2}$
4. $c_2[(a - d) \cos \alpha + b \sin \alpha + d] + [d \cos \alpha - (a - d) \sin \alpha]\sqrt{c_1^2 - c_2^2}$

TABLE 2

BOUNDARY EQUATIONS BETWEEN ADJACENT PATHS

Sector I
1/2. $y = x \frac{(\sin \alpha + m \cos \alpha)}{(\cos \alpha - m \sin \alpha)}$

Sector II
1/2. $y = x \frac{(\sin \alpha - m \cos \alpha)}{(\cos \alpha + m \sin \alpha)}$
1/3. $y = x \frac{\cos \alpha + \sin \alpha - 1}{(1 + \cos \alpha)m - \sin \alpha}$
2/3. $y = xm$
3/4. $y = (x - d) \frac{\cos \alpha - m \sin \alpha - 1}{(1 - \cos \alpha)m - \sin \alpha}$
4/1. $y = x \tan - \frac{d}{2m} (m \tan \alpha - 1 + \sec \alpha)$

The graphical representations were then used to project certain characteristics of the network and hinterland:

1. With reference to Figure 7 (a, b & c), as $C_1$ is increased with $m$ increasing ceteris paribus, the development of a hinterland pattern for each secondary route is such that all the area in a sector is equally portioned out to the roads around it.

2. For a fixed size of hinterland, as $\alpha$ decreases the lengths of the feeder lines increase. Conversely, as $\alpha$ decreases, the cost of construction increases since it is a function of the length of the feeder lines [Figure 6(a), 7(a), 8(a) and 9(a)].

3. For all $\tan^{-1}m = \theta < \alpha/2$ the movement zones change radically, and no longer do all the flow cost functions hold. The secondary routes are no longer significant determinants of the flow pattern or cost. Apart
from the first pair, the existence of all remaining pairs is incidental
to the overall flow pattern [Figure 6(c)].

4. In Figures 8(a) and 9(a) and (b), the boundaries become irregular.
Interpreted another way, if the original boundary is adhered to, "the
optimum slope" rule is no longer applicable at certain portions of the
edge. The phenomenon only occurs if \(\tan^{-1}m = \theta > \alpha\).

**Minimum Total Flow Cost**

If the total flow cost for the network is evaluated for a specific area, the
significance of each of the parameters can be tested by varying one while
keeping all others fixed. Here the classical method of integrating each of the
cost functions between their respective limits in the appropriate sectors will
be used to obtain the total flow cost.

Derivation of a general expression for the total flow cost is feasible within the
context of certain assumptions. From Figure 5, it can be noted that the
minimum paths in Sectors II and III are alike except for the straight line
path to the focus which exists in Sector II. If the analysis is restricted such
that \(\tan^{-1}m \geq \alpha\), Sectors II and III become identical, thereby reducing the
total number of atypical or incomplete sectors. Thus, for \(\alpha/2 \geq \tan^{-1}m \geq \alpha\),
the following expression represents the total flow cost.

\[
T.F.C. = F_{1c} + F_{2c} + A_1 \cdot d \cdot c_2 + F_{3c} + A_4 \cdot d \cdot c_2
+ F_4 + \cdots + A_{(n-1)} (n - 3) d \cdot c_2 + F_{(n-1)c} - F_{nc} + A_n \cdot (n - 2) d \cdot c_2
\]

where:
- \(F_{ic}\) = flow cost of the \(i\)th sector, \(i = 1 \cdots n\)
- \(A_i\) = area of the \(i\)th sector
- \(d\) = interval between feeder lines
- \(c_2\) = cost of travelling on the network

Since \(F_{2c} = F_{3c} = \cdots = F_{(n-d)c}\)

\[
T.F.C. = 2 \left\{ F_{1c} + \sum_{i=2}^{(n-1)} F_{ic} + d \cdot c_2 \sum_{i=2}^{(n-1)} (i - 2) A_i + (N - 2) d \cdot c_2 A_n + F_{nc}\right\}
\]

Intuitively, the existence of an optimum \(\alpha\) for a range of values of \(m\) and \(d\)
is apparent, and an attempt was made to locate this value. To facilitate inte-
gration of the cost functions over their limits, an \(\alpha^o\) coordinate system was
adopted.\(^{1}\) The converted functions and boundaries of the various sectors are

\(^{1}\) By definition it is a coordinate system with an angle \(\alpha\) between the abscissa and the
ordinate. \(y', x'\) represent the axis of the \(\alpha\) coordinate system, where: \(y = y' \sin \alpha\) and
\(x = x' + y' \cos \alpha\).
TABLE 3

MINIMUM COST FUNCTIONS OF FLOW PATHS BASED ON $\alpha^\circ$ CO-ORDINATE SYSTEM

Sector I

1. $c_1\sqrt{x^2 + y^2} + 2xy \cos \alpha$
2. $x(c_2 \cos \alpha - \sin \alpha \sqrt{c_1^2 - c_2^2}) + c_2 y$

Sector II

1. $x(c_2 \cos \alpha + \sin \alpha \sqrt{c_1^2 - c_2^2}) + c_2 y$
2. $c_1 \sqrt{x^2 + y^2} + 2xy \cos \alpha$
3. $c_2 x + y(c_2 \cos \alpha + \sin \alpha \sqrt{c_1^2 - c_2^2})$
4. $x(c_2 \cos \alpha - \sin \alpha \sqrt{c_1^2 - c_2^2}) + c_2 y + d(c_1 - c_2 \cos \alpha + \sin \alpha \sqrt{c_1^2 - c_2^2})$

TABLE 4

BOUNDARY BETWEEN THE MINIMUM PATHS IN SECTORS I AND II

Sector I

1/2. $y = \frac{x(\sin \alpha + m \cos \alpha)}{m}$

Sector II

1/2. $y = \frac{x(\sin \alpha - m \cos \alpha)}{m}$

2/3. $y = \frac{xm}{\sin \alpha - m \cos \alpha}$

1/3. when $m = \frac{\alpha}{2}$, $y = x$

3/4. $y = \frac{(x - d) (\cos \alpha - m \cos \alpha - 1)}{(\cos \alpha + m \sin \alpha - 1)}$

4/1. $x = \frac{d}{2m} (m - \cot \alpha + \cosec \alpha)$

tabulated in Tables 3 and 4. An attempt was made to locate an optimum by solving $dF(\alpha)/d\alpha = 0$, where $F(\alpha)$ is the total flow cost. However, since the resulting 6th degree differential equation cannot be readily solved, this approach was abandoned. Instead, as a substitution for the global optimum, $F(\alpha)$ was evaluated for various values of $\alpha$ to obtain an approximate value for the optimum angle.

As an alternative method, the total flow cost could have been evaluated by the summation of the cost functions for a random set of points, adjusted by a multiplier factor. However, this would have involved making a qualitative decision on the number of points needed to reduce the margin of error. Also, the confidence level of a given number of points used can only be crudely estimated. One can therefore conclude that integration, though laborious, is the most accurate and exact method. As a method, its use can be questioned only if the input of effort outweighs the significance of the outcome.
SENSITIVITY ANALYSIS

A test of significance is attempted for each parameter by allowing each one to vary from one extreme value to the other, while maintaining others constant. The following discussion is focused upon the sensitivity of each parameter and the identification of the optimum values of each, where the optimal values are defined as those which will give the minimum total flow cost.

m: the Optimum Slope.—The parameter m can have any value from 0, to $+\infty$. When $m = 0$, then the ratio of $c_1:c_2$ is 1:1. However, this would violate the basic assumption that $c_1 > c_2$. $m = +\infty$ if $c_1 = 0$ or, if interpreted in another way $\theta = 90^\circ$. If the optimum slope is $90^\circ$ then the minimum flow cost path from any point is composed of a perpendicular line to the nearest branch of the network and of the remaining distance covered along the network to the focus. Since a cost is incurred only for the distance traveled off the network it is intuitively obvious that the optimum value of $m$ is $+\infty$ (Figure 10).

However, in reality, the existence of such a condition is highly improbable. Through network improvements can minimize $c_2$, the flow cost on the network, it can never be reduced to zero.

d: the Spacing Between Secondary Routes.—The parameter d can logically vary between 0 and $+\infty$ though only the lower extreme is meaningful. If the optimal value of d is 0 then the total area of the model is covered completely by secondary routes. However, it may be recalled that flow cost has been defined as a function of construction cost, so, unless the construction cost is 0, the optimum value of d must be greater than zero.

$\alpha$: the Angle Between a Secondary Route and the Trunk Line.—To test the significance of a variable $\alpha$, $F(\alpha)$ was evaluated for $\alpha = 30^\circ$, $45^\circ$, $60^\circ$, $75^\circ$, and $90^\circ$ at the constant values $b = 1$, $d = 1$, $a = 5$, and $m = 1/\sqrt{3}$. The values were then plotted against $\alpha$. From the graph (Figure 11) a number of significant conclusions pertaining to the whole framework can be drawn:

1. There is a break in the curve at $\alpha = 30^\circ$. For all values of $\alpha < 30^\circ$, the integrable area is not coincident with the rectangle and poses severe problems.

![Fig. 10. Direction of Traffic Flow: Optimum Value of m.](image-url)
of error propagation. Thus, the curve can be extrapolated for $\alpha < 30^\circ$, by speculation only. However, this does not provide a major setback as the optimum angle by definition located at the minimum value of $F(\alpha)$ does not lie within this range of values for $\alpha$.

2. A second break exists at $\alpha = 60^\circ$, when the minimum flow pattern changes radically. All pairs of secondary routes or feeder lines, apart from the first, have no value and are incidental to the pattern. Thus, the effective network changes as indicated previously.

3. Approximate location of the optimal angle $\alpha$, is $60^\circ < \alpha < 75^\circ$. The optimal angle thus located indicates the inefficiency of the initial pattern selected for the model. As a solution to this model a modified initial form has emerged as the optimal network configuration where only the addition of a pair of secondary routes to the trunk lines is mandatory. This somewhat unexpected result can be attributed to the values given to the constant parameters. More specifically, the location of $\alpha$ is directly related to $m$ evaluated at $1/\sqrt{3}$. It is recalled that $m$, is equivalent to $\tan \theta$ or $\sqrt{(c_1^2 - c_2^2)/c_2^2}$. Thus, at $m = 1/\sqrt{3}$, the ratio between $c_1:c_2$ is 1:0.865, so the benefit obtained from traveling on the network relative to traveling off the network is minimal. Thus, the changed pattern illustrates preference for a shorter route to the focus over any other indirect, least-cost paths.

However, if the analysis is constrained such that $\alpha/2 < \tan^{-1}m < \alpha$, then
the optimal angle for the given pattern is $60^\circ$ for the specific case evaluated. In general, the optimal angle for a "dendritic" pattern of feeder lines is $\tan^{-1}2m$ for a variable $m$. This is the only global suboptimum solution which can be extrapolated from the results.

The obvious shortcomings of the above analysis are that a ratio of $1:0.865$ for $c_1:c_2$ is not realistic. So any attempt to draw an analogy between these results and any real world case must necessarily contain inherent deficiencies. However, it was necessary to select this ratio to avoid problems of changed boundaries encountered for $m > \tan \alpha$ as shown in 8(a), 9(a) and 9(b).

All Other Parameters.—The remaining parameters, $a$, the length of the rectangle, and $b$, the width of the rectangle are interrelated. Both can be varied between the values 0 and $\infty$ with complementary variations, in the cost of construction and in the minimum total flow cost. At either extreme, for any one of the two parameters, the basic prerequisite of a bounded area no longer exists. This precludes the use of the criterion of minimum total flow cost for an optimum value of $a$ or $b$. However, despite this, the interrelationship between the lengths of the trunk line and the secondary routes (a function of $b$) is worth noting. Substitution between $a$ and $b$ can readily take place such that the area is constant without violating any assumptions or changing any other parameter. However it is intuitively obvious that the total flow cost will vary. A corollary to this is that the location of the focus is significant, even if it is restricted to the center of a side.

**SUMMARY**

The methodology applied to this city-hinterland case did not yield results that could be generalized easily since even in this limited case the combinatorial difficulties encountered forced a specific analysis. However this study has indicated the relative significance of the parameters, defining the model within a co-ordinate system. These parameters are: the width and the breadth of the rectangle, the optimum angle at which the traffic flow that is off the network enters the network, the spacing interval between the parallel secondary routes branching out from the main trunk line, and the angle between the secondary routes and the main line. If all alternative geometric networks could be defined by similar sets of parameters, a general method of analysis could be developed.

One of the goals of the study was to identify the unique optimal value for the angle between the main and the feeder routes, while minimizing the total flow cost over the area. The flow cost is a function of the transport costs per unit distance, off and on the network. In this respect the study is incomplete, since computationally no method as yet lends itself to this solution. As a substitute, the general function defining the total minimum flow cost over the network was plotted against different values of this angle, while all other parameters were evaluated as constants. However, the resulting curve only yielded an approximate solution. Of the remaining parameters, only the optimum slope at which all flows approach the network and the spacing
interval between the parallel secondary routes are of major significance to the minimization of flow cost. The derivation of optimum values for both of these parameters only required a logical extension of the functions describing their geometrical relationship to the model.

The fact that only three parameters are significant in terms of sensitivity suggests that the model can be calibrated better if the insignificant variables are not defined explicitly but rather incorporated implicitly in other, possibly more relevant variables, e.g., differential capacity between trunk and feeder lines and differentiation of incoming and outgoing traffic flow from the focus. These modifications will enhance the validity of a comparison between the model results and a real-world situation.

The results of this study have indicated two interrelated areas in which further work must be carried out. First, tests of comparable networks should be made to help identify other parameters which influence the sensitivity of the cost functions in network geometry. Second, alternative algorithms, such as the calculus of variations, must be tested to overcome the combinatorial complexities. Only then will the possibility of identifying an optimal network geometry for a set of specified goals be within reach.

While the major concern here has been with network geometry, in order to achieve an optimal network design without any major flaws, topology must be brought explicitly within the model framework. Although the topologic factor was implicit in this city-hinterland case, at this stage of our knowledge of network geometry connectivity considerations are not of paramount importance in the group of problems classified under optimal transport network designs.

**LITERATURE CITED**


